THE HARPUR EUCLID

The Harpur Euclid

AN EDITION OF

EUCLID'S ELEMENTS

REVISED IN ACCORDANCE WITH THE REPORTS OF THE CAMBRIDGE BOARD OF MATHEMATICAL STUDIES, AND THE OXFORD BOARD OF THE FACULTY OF NATURAL SCIENCE

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PREFACE

THIS work is intended to be strictly a School Edition of Euclid's Elements. While retaining his sequence of propositions, and basing their proofs entirely on his axioms, the Editors have not scrupled to replace some of his demonstrations by easier ones, and to discard much superfluous matter, irritating alike to student, teacher, and examiner, from those retained.

Symbols have been introduced at an early stage, but their use has been avoided in the first eight propositions, in order that beginners might not be confronted with two difficulties at once.

An attempt has been made in the Notes and Exercises to familiarise the student with such terms and ideas as he will be likely to meet with in his higher reading and in treatises on Elementary Geometry by other writers, and to indicate by difference of type important theorems and problems which should be well known to him although not given among Euclid's propositions. At the same time the work is

not intended to be a substitute for such works as Casey's well-known *Sequel to Euclid*, but to serve as an introduction to them.

The Editors are of opinion that students of Geometry cannot attempt the solution of Geometrical Exercises too soon, and they have attached simple riders to most of the propositions which ought to be within the reach of all those who have mastered the book-work intelligently.

Much use has been made of the excellent *Syllabus of Plane Geometry* issued by the Association for the Improvement of Geometrical Teaching, and readers of Professor Henrici's remarkable little work on *Congruent Figures* will probably detect its influence throughout.

The short treatise 'On Quadrilaterals' will, it is hoped, be found interesting to those students who have mastered the previous exercises, and useful to teachers in supplying a large number of easy and instructive exercises in a short compass.

The final collection of Miscellaneous Exercises is purposely taken from widely different sources; some are—as far as it is safe to make such an assertion of any proposition in Elementary Geometry—original others have been taken from or suggested by various examination papers, and others are well known theo-

rems or problems given by most writers on the same subject.

The Editors have to thank several friends for examining proof-sheets; in particular, Mr. A. F. Smith, B.A., The Grammar-School, Chester; Mr. H. Crofts, B.A., Parkfield School, Liverpool; Mr. G. M. Bates, B.A., Headmaster of the Harpur Trust Elementary School; and Mr. A. E. Field, B.A., of Trinity College, Oxford. They would be obliged to those teachers who use their work as a class-book for early notice of any errors that have escaped correction, and for such suggestions for its improvement as experience may provide.

EDWARD M. LANGLEY.
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NOTE.

THE Editors desire to call attention to the Reports of the Mathematical Board at Cambridge, and the Board of the Faculty of Natural Science at Oxford, upon the Memorial presented by the Council of the Association for the improvement of Geometrical Teaching to these Universities, praying for such changes in their Examinations in Elementary Geometry 'as would admit of the subject being studied from Text-books other than editions of Euclid, without the student being thereby placed at a disadvantage in those Examinations.'

The Cambridge Board reports :—

> 'The majority of the Board are of opinion that the rigid adherence to Euclid's texts is prejudicial to the interests of education, and that greater freedom in the method of teaching Geometry is desirable. As it appears that this greater freedom cannot be attained while a knowledge of Euclid's text is insisted upon in the examinations of the University, they consider that such alterations should be made in the regulations of the examinations as to admit other proofs besides those of Euclid, while following however his general sequence of propositions, so that no proof of any proposition occurring in Euclid should be accepted in which a subsequent proposition in Euclid's order is assumed.'

The Oxford Board, in nearly equivalent terms, reports :—

> '1. That a rigid adherence to the ordinary text-books of Euclid should no longer be insisted on, but that a greater freedom of demonstration should be allowed, both in Geometrical teaching and in Examination.
>
> '2. That, nevertheless, Euclid's *method* should be required in all Pass Examinations in Geometry, in so far as that no axioms other than those of Euclid shall be admitted, and that no proof of a proposition be allowed which assumes the truth of any proposition which does not precede it according to Euclid's order.'

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DEFINITIONS.

Of the thirty-five definitions with which it has been usual to preface Book I., only about one-third part need be mastered by the student before he attempts the propositions. These are given below. Of the rest, such as he will want for Book I. are given in due course. For examination purposes and for reference, a complete list of Definitions, Axioms, and Postulates is given on pp. 96-98.

The 'Syllabus,' to which occasional references are made, is that published by the Association for the Improvement of Geometrical Teaching.

1. A point is that which has no parts, or which has no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

A straight line is sometimes spoken of as the **join** *of its extreme points.*

5. A superficies (or surface) is that which has only length and breadth.

6. The extremities of a surface are lines.

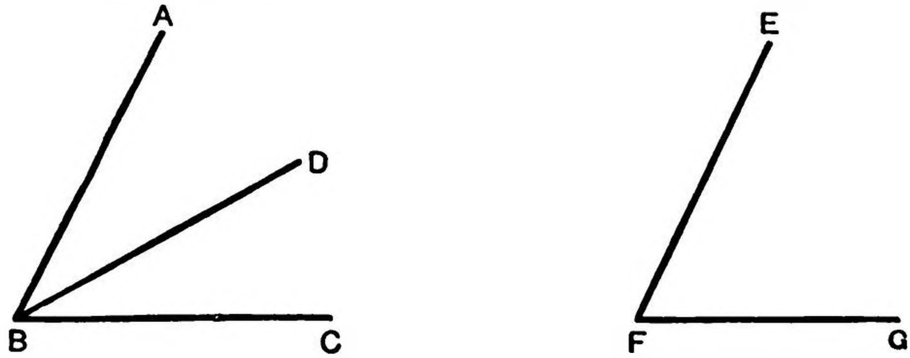
7. A plane superficies (flat surface) is that in which any two points being taken, the straight line between them lies wholly in that superficies.

9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

It is pointed out in the Syllabus that the term angle is incapable of real definition. It would be better to say with the Syllabus, **'When two straight lines are drawn from the same point they are said to contain, or to make with each other, a plane angle,'** *and to indicate the nature of an angle by saying that the angle is greater or less, according as the amount of turning round the point that would bring either of the lines into coincidence with the other is greater or less.* (See Syllabus.)

NOTE.

When several angles are at one point B, *any one of them is expressed by three letters, of which the letter which is at the vertex of the angle, that is, at the point at which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus the angle which is contained by the straight lines* AB, CB *is*



named the angle ABC *or* CBA; *the angle which is contained by the straigh lines* AB, DB *is named the angle* ABD *or* DBA; *and the angle contained by the straight lines* DB, CB *is named the angle* DBC *or* CBD. *But if there is only one angle at a point, it may be expressed by a letter placed at that point; thus the angle* EFG *might be called simply the angle* F.

The latter method should always be employed when possible.

14. A figure is that which is enclosed by one or more boundaries.

A figure contained by **three straight lines** *is called a* **triangle.**

15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal.

16. This point is called the centre of the circle, and the straight line drawn from the centre to the circumference is called the radius of the circle.

POSTULATES.

Let it be granted

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. That a circle may be described from any centre at any distance from that centre.

In practice this amounts to demanding the use of a **ruler** *or* **straight-edge** *and a* **pair of compasses.**

Neither of these instruments, however, is to be used for carrying distances [see note 2 on Prop. 2].

When Geometers speak of a problem such as **to divide a given angle into three equal angles** *as 'impossible,' they mean 'impossible under the restrictions they have placed upon themselves as to the elementary constructions they take for granted.'*

They only take for granted the three 'demanded' by the Postulates.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

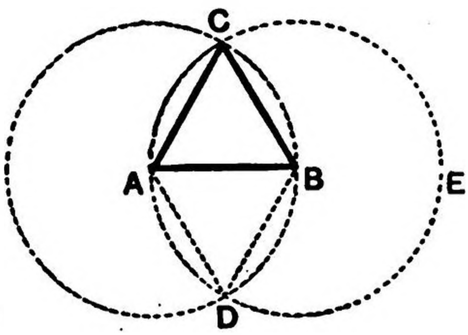
11. All right angles are equal to one another.

Def.—A triangle with its three sides equal to each other is called an Equilateral Triangle.

PROPOSITION 1. Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line; it is required to describe an equilateral triangle on AB.



From the centre A, at the distance AB, describe the circle BCD.
From the centre B, at the distance BA, describe the circle ACE. [Post. 3.

From the point C, at which the circles cut one another, draw the straight lines CA, CB to the points A and B.

Because the point A is the centre of the circle BCD, therefore AC is equal to AB; [Def. 15.

And because the point B is the centre of the circle ACE, therefore BC is equal to BA; [Def. 15.

therefore AC, BC, AB are all equal to one another; [Ax. 1.

therefore ABC is equilateral.

NOTES.

1. The words 'join CA' are afterwards used as an abbreviation for 'draw a straight line from C to A.'

2. If we join AD and BD (D being the other point in which the two circles cut each other) another equilateral triangle ADB will be described on the given straight line AB.

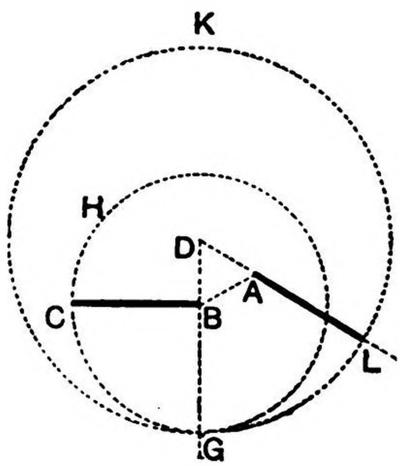
3. 'A triangle is sometimes regarded as standing on a selected side, which is then called its **base**, and the intersection of the other two sides is called the **vertex**.' (Syllabus.)

4. The **quadrilateral** (four-sided) figure ACBD has **all its sides** equal. Such a quadrilateral is called a **rhombus**.

PROPOSITION 2. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw from the point A a straight line equal to BC.



Join AB. [POST. 1.

On AB describe the equilateral triangle DAB. [I. 1.

From the centre B, at the distance BC, describe the circle CGH cutting DB produced in G. [POST. 3.

From the centre D, at the distance DG, describe the circle GKL cutting DA produced in L. [POST. 3.

Then AL shall be equal to BC.

Because D is the centre of the circle GKL, therefore DL is equal to DG; [DEF. 15.

But DA is equal to DB, [CONST.
therefore the remainder AL is equal to the remainder BG. [AX. 3.
Because B is the centre of the circle CGH,
therefore BC is equal to BG; [DEF. 15.
But it has been shown that AL is equal to BG;

therefore AL is equal to BC. [AX. 1.

NOTES.

1. (*a*) The point A may be joined with either end of the line CB;

(*b*) The equilateral triangle described on this join as base may be described on either side of that base;

(*c*) The other two sides of the equilateral triangle may be produced either both through the vertex, or, as in the diagram, both in the opposite direction; so that with BC and A given there are eight ways in which the problem might be solved. The student should try to go through the other seven by himself.

2. The demonstration of the possibility of solving this problem has been rendered necessary by the restriction which Euclid has placed upon the interpretation of his Third Postulate. If instead of this postulate we assumed that of the Syllabus,

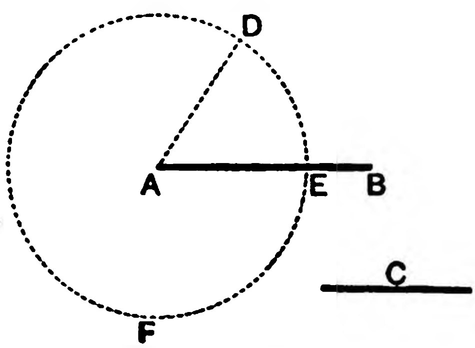
'A circle may be described from any centre with a radius equal to any finite straight line,'

this proposition and the next would be unnecessary. Euclid only uses his postulate as if it allowed him to describe a circle from any centre and with any straight line drawn from that point as radius.

PROPOSITION 3. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater; it is required to cut off from AB, the greater, a part equal to C, the less.



From the point A draw the straight line AD equal to C. [I. 2.

From the centre A, at the distance AD, describe the circle DEF cutting AB at E. [POST. 3.

AE shall be equal to C.

Because A is the centre of the circle DEF,
therefore AE is equal to AD; [DEF. 15.

But C is equal to AD; [CONST.

therefore AE is equal to C. [AX. 1.

NOTES.

If the student finds much difficulty in mastering the demonstrations of Propositions 4, 5, 6, 7, he would probably do better *to take their results for granted and go on with the easier ones which follow*, coming back from time to time to those which he has found too difficult.

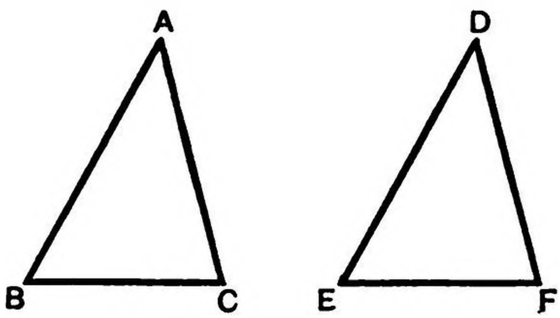
By the time he has reached Proposition 16 he ought no longer to find them a stumbling-block.

The course taken by a teacher must naturally depend upon the intelligence of his pupils and their interest in the subject, but it will probably be found best, after explaining the general and particular enunciations of the theorems mentioned, to pass on in the way just suggested. After practice in the easier demonstrations which follow, they will be better able to grapple with the difficulties with which they were previously unable to cope.

PROPOSITION 4. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and the angle BAC equal to the angle EDF: the base BC shall be equal to the base EF, and the triangle ABC to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite, namely the angle ABC to the angle DEF, and the angle ACB to the angle DFE.



For if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB on the straight line DE, the point B will coincide with the point E, because AB is equal to DE; [HYP.

And AC will fall on DF, because the angle BAC is equal to the angle EDF; [HYP.

therefore also the point C will coincide with the point F, because AC is equal to DF; [HYP.

But the point B was shown to coincide with the point E;
therefore the base BC will coincide with the base EF,
for if not, two straight lines would enclose a space, which is impossible; [AX. 10.
therefore the base BC is equal to the base EF; [AX. 8.
and the whole triangle ABC coincides with the whole triangle DEF, and is equal to it. [AX. 8.

Also since AB, BC coincide with DE, EF,
therefore the angle B is equal to the angle E. [AX. 8.

And since AC, CB coincide with DF, FE,

therefore the angle C is equal to the angle F. [AX. 8.

NOTES.

1. The method of proof here used is called **Superposition.**

Though often found difficult by a beginner, it is very valuable, and should be practised whenever possible.

It is the most elementary of all methods. Euclid, however, has not availed himself of it as much as he might have done.

2. 'Figures that may be made by superposition to coincide with one another are said to be identically equal; and every part of one being equal to a corresponding part of the other, they are said to be equal in all respects.' (Syllabus.)

3. Professor Henrici calls such figures **congruent.**

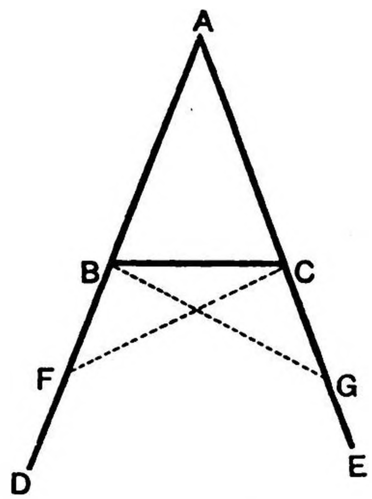
Ex. 1.—Two quadrilaterals, ABCD and PQRS, have the sides AB, BC, CD respectively equal to the sides PQ, QR, RS, and the angles B and C respectively equal to the angles Q and R. Show that the quadrilaterals are congruent.

DEF.—**A triangle with two of its sides equal to each other is called an 'isosceles' triangle.**

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal to one another, and, if the equal sides be produced, the angles on the other side of the base shall be equal to one another.

Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E, the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.



In BD take any point F, and from AE, the greater, cut off AG equal to AF, the less, [I. 3.
and join FC, GB.

Because AF is equal to AG (const.) and AB to AC, [HYP.
the two sides FA, AC are equal to the two sides GA, AB, each to each,
and they contain the angle FAG common to the two triangles AFC, AGB,
therefore the base FC is equal to the base GB, the angle ACF is equal to the angle ABG, and the angle AFC is equal to the angle AGB. [I. 4.

Again, because the whole AF is equal to the whole AG, [CONST.
and the part AB is equal to the part AC, [HYP.
therefore the remainder BF is equal to the remainder CG. [AX. 3.

Now, FC has been shown to be equal to GB,
and the angle BFC (contained by BF, FC) to the angle CGB (contained by CG, GB);
therefore the angle CBF is equal to the angle BCG, [I. 4.
and the angle BCF is equal to the angle CBG. [I. 4.

But the whole angle ACF has been shown to be equal to the whole angle ABG,
as well as the part BCF to the part CBG;

therefore the remaining angle ACB is equal to the remaining angle ABC. [AX. 3.

COROLLARY.—**Hence every equilateral triangle is also equiangular.**

A **corollary** is a proposition, the truth of which follows immediately or may easily be deduced from what has been demonstrated in the proposition to which it is attached.

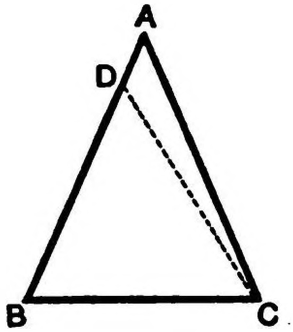
I. 5 is such a traditional stumbling block to beginners that it has become known as the 'pons asinorum.' Another epithet seems at one time to have been attached to it:—

'Quinta propositio geometriæ Euclidis dicitur "Elefuga," id est, fuga miserorum.'—ROGER BACON (*Opus tertium*, Cap. 6).

PROPOSITION 6. THEOREM.

If two angles of a triangle be equal, the sides also which subtend, or are opposite to the equal angles, shall be equal to one another.

Let ABC be a triangle, having the angle ABC equal to the angle ACB: the side AC shall be equal to the side AB.



For if AC be not equal to AB, one of them must be greater than the other.

Let AB be the greater, and from it cut off BD, equal to AC the less, [I. 3.
and join DC.

Then in the two triangles DBC, ACB,
we have DB equal to AC, [CONST.
BC common,
and the angle DBC (contained by DB, BC) equal to the angle ACB (contained by AC, CB); [HYP.
therefore the triangle DBC is equal to the triangle ACB, [I. 4.
the less to the greater, which is absurd; [AX. 9.
therefore AB is not unequal to AC,
that is, AB is equal to AC.

COROLLARY.—**Hence every equiangular triangle is also equilateral.**

NOTE.

A theorem consists of two parts, the **hypothesis**, or that which is assumed, and the **conclusion**, or that which is asserted to follow therefrom.

Two theorems are said to be **converse**, each of the other, when the hypothesis of each is the conclusion of the other (Syllabus).

Thus each of the two theorems 5 and 6 is the converse of the other.

The student should notice that the converse of a true theorem may or may not be true.

Ex. 2.—In the figure of Prop. 5, if H is the point where BG cuts CF, BH is equal to HC.

Also FH is equal to HG.

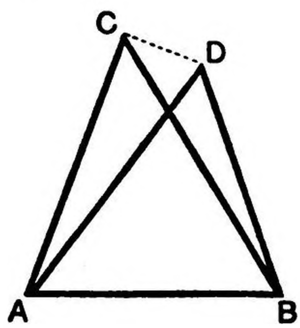
Ex. 2 (*a*).—The sides AB, AC of a triangle ABC are produced to D and E. If the angle DBC is equal to the angle ECB, then the side AB is equal to the side AC (*Clavius*).

In BD *take any point* F. *From* CE *cut off* CG *equal to* BF. *Join* BG, CF, FG. *Show that I.* 4 *applies* (i) *to triangles* FBC, GCB (ii) *to triangles* FBG, GCF *and hence, by I.* 6, *that* AF *is equal to* AG.

PROPOSITION 7. THEOREM.

On the same base and on the same side of it there cannot be two triangles having their sides, which are terminated in one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

On the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides CA, DA, which are terminated at A, equal to one another, then they cannot have also the sides CB, DB, which are terminated at B, equal to one another.



Join CD.

In the case in which the vertex of each triangle is without the other triangle;

Because AC is equal to AD, [HYP.

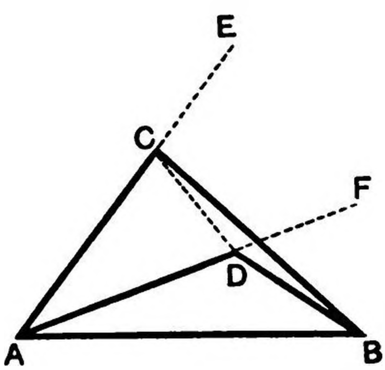
therefore the angle ACD is equal to the angle ADC; [I. 5.

But the angle ACD is greater than the angle BCD, [AX. 9.

therefore the angle ADC is also greater than the angle BCD;

much more then is the angle BDC greater than the angle BCD,

therefore BC is not equal to BD;* [I. 5.



But if one of the vertices, as D, be within the other triangle ACB,

produce AC, AD to E and F respectively.

Then because AC is equal to AD, [HYP.

therefore the angle ECD is equal to the angle FDC; [I. 5.

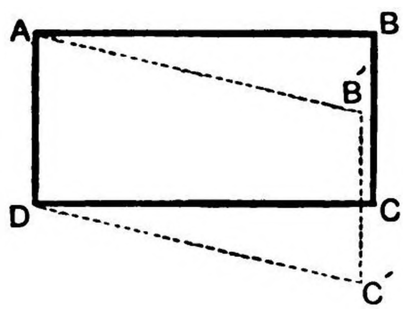
But the angle ECD is greater than the angle BCD, [Ax. 9.
therefore the angle FDC is also greater than the angle BCD;
much more then is the angle BDC greater than the angle BCD,

therefore BC is not equal to BD.* [I. 5.

The case in which the vertex of one triangle is on a side of the other needs no demonstration.

* If the student has any difficulty in seeing that this follows from I. 5, he should insert this line :

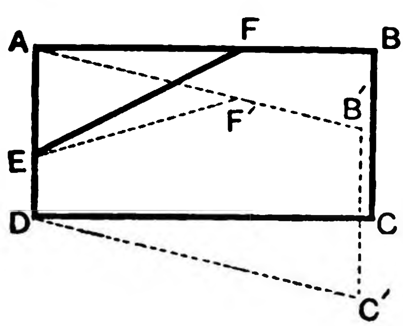
For if it were, the angle BDC would be equal to the angle BCD.



I. 7 is the foundation of the well-known practical device for securing rigidity in the framework of a gate.

The framework ABCD could, if there was looseness at the joints, easily be distorted into the shape shown by AB'C'D without having to stretch or compress any of the bars of which it is composed.

But if a fifth bar, EF, were pivoted to AB, AD at E and F, it would be impossible for the distortion indicated in the figure to take place so long as AF, EF remained of the same

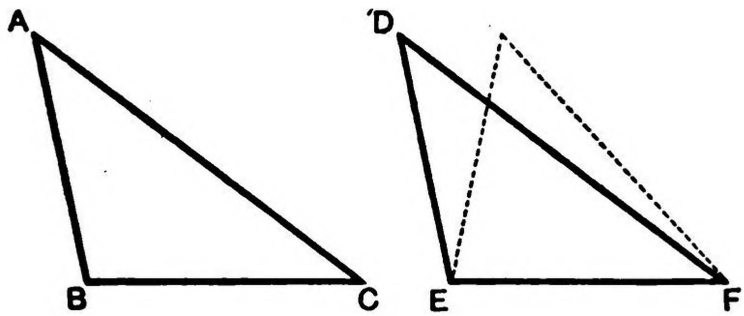


length, otherwise there could be two triangles AFE, AF'E, on the same base AE, and on the same side of it, having AF equal to AF' and EF equal to EF'.

PROPOSITION 8. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC eqaal to the two sides DE, DF each to each, namely AB to DE, and AC to DF, and also the base BC equal to the base EF; the angle BAC shall be equal to the angle EDF.



For if the triangle ABC be applied to the triangle DEF, so that the point B may be on the point E, and the straight line BC on the straight line EF, the point C will also coincide with the point F, because BC is equal to EF; [HYP.

Therefore BC coinciding with EF, BA and AC will coincide with ED and DF.

For if not, on EF as base there will be another triangle formed on the same side of it as DEF, and having the side terminated at E equal to DE,
and also the side terminated at F equal to DF,
which is impossible; [I. 7.

therefore the angle BAC coincides with the angle EDF, and is equal to it.

NOTE.

We have really demonstrated more than the enunciation asserts; for we have shown that the two triangles are *congruent*. The student would do well to learn the enunciation in the subjoined form :—

If two triangles have the three sides of the one equal to the three sides of the other, each to each, then the triangles are identically equal, and of the angles, those are equal which are opposite to equal sides. (Syllabus.)

Ex. 3.—In the figure of Prop. 1 join AD, BD, and prove that the angles CAB, DAB are equal to each other.

Ex. 4.—In the figure of Prop. 1 join CD and prove that the angles ACD, BCD are equal to each other.

Ex. 5.—In the figure of Prop. 1 let E be the point where CD cuts AB. Then AE is equal to EB.

Ex. 6.—If ABCD is a rhombus (that is, a quadrilateral having all its sides equal to one another), and AC cuts BD in E, then AE is equal to EC, and BE is equal to ED; also the four angles AEB, BEC, CED, DEA are all equal to one another.

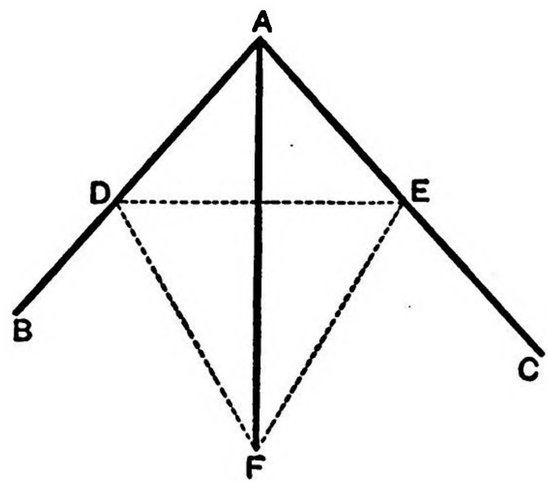
From I. 9 onwards, certain symbols will be used as abbreviations for words which occur frequently :—

$\because$	will be used	for	'because.'
$\therefore$	,,	,,	'therefore.'
$=$	,,	,,	'is equal to,' or 'are equal to.'
$>$	,,	,,	'is greater than,' or 'are greater than.'
$<$	,,	,,	'is less than,' or 'are less than.'
$\angle$	,,	,,	'angle.'
$\triangle$	,,	,,	'triangle.'
$\odot$	,,	,,	'circle.'

PROPOSITION 9. PROBLEM.

To bisect a given rectilineal angle—that is, to divide it into two equal parts.

Let BAC be the given rectilineal angle: it is required to bisect it.



Take any point D in AB and from AC cut off AE equal to AD. [I. 3.

Join DE.

Upon DE, on the side remote from A, describe the equilateral △ DEF. [I. 1.

Join AF.

The straight line AF shall bisect ∠ BAC.

In the two △s FAD, FAE

AD = AE, [CONST.

AF is common,

and DF = EF; [DEF. 24.

∴ ∠ DAF = ∠ EAF. [I. 8.

NOTES.

1. The student should satisfy himself as to the necessity of the words *on the side remote from* A.

2. The demonstration would be just as easy if the △ DFE were merely an isosceles △ having DE for its base; and in Practical Geometry, after cutting off AE equal to AD (by describing an arc with any radius from centre A), arcs with any equal radii long enough to ensure their cutting are described with centres D and E, and the point F, where they cut, is joined with A.

The student may perhaps find an indication of what may have been Euclid's reason for choosing his method instead of the more general 'practical' one in Note 2 on Prop. 2.

3. A quadrilateral like ADFE (having AD, DF equal to AE, EF respectively) is called a **Kite.**

Note that it consists of two congruent △s, ADF, AEF, on opposite sides of a common base, AF, and such that superposition could be effected by turning either of them about that common base until it came into the plane containing the other.

From this property it is called **a symmetrical** figure, and AF is called **its axis of symmetry.**

Ex. 6 (*a*).—If in the fig. of I. 9 an equilateral △ DGE be described on the same side of DE as A and AG be joined, show that ∠ DAG = ∠ EAG.

Ex. 6 (*b*).—If in the fig. of I. 9 a pt. H be taken on AF, or AF produced, show that HD = HE, and ∠ HDF = ∠ HEF.

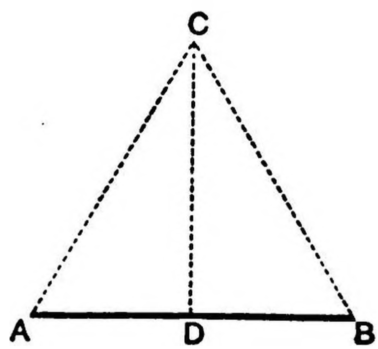
Ex. 6 (*c*).—The sides OA, OB of a △ OAB are equal to each other, and the angles OAB, OBA are bisected by st. lines AP, BP meeting in P. Show that PA = PB.

Ex. 6 (*d*).—The equal sides OA, OB of an isosceles △ OAB, are produced to C and D respectively, and the angles CAB, DAB bisected by st. lines AP, BP meeting in P. Show that PA = PB.

PROPOSITION 10. PROBLEM.

To bisect a given finite straight line—that is, to divide it into two equal parts.

Let AB be the given straight line; it is required to divide it into two equal parts.



Describe on AB an equilateral △ ABC. [I. 1.

Bisect ∠ ACB by the straight line CD, meeting AB at D. [I. 9.

AB shall be divided into two equal parts at the point D.

In the two △s ACD, BCD

AC=CB, [CONST.

CD is common,

and ∠ ACD= ∠ BCD, [CONST.

∴ AD=DB. [I. 4.

NOTE.

The student should go through the successive steps practically, and compare Euclid's process with that given in Manuals on Practical Geometry.

He will find very little difference between the two, and should satisfy himself of the correctness of the 'practical' method. He should also find a reason for Euclid having chosen the method of Prop. 10 instead of it. See Note 2 on Prop. 9.

Ex. 6 (*e*).—If in the fig. of I. 10 a pt. H be taken or DC, or DC produced, show that HA = HB, and that ∠ HAC = HBC.

Ex. 6 (*f*).—The equal sides OA, OB of an isosceles △ OAB are bisected at C and D respectively. Show that if CD be joined ∠ OCD = ∠ ODC.

DEF. 10.—**When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is said to be perpendicular, or at right angles, to it.**

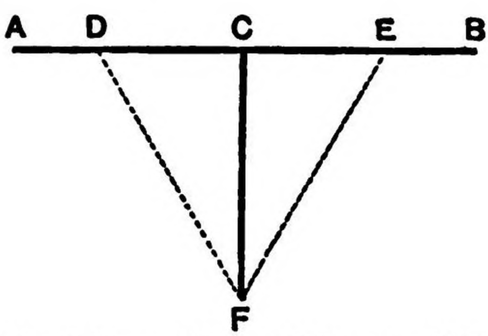
The symbol for 'perpendicular' is ⊥r.

The student need draw no distinction between the terms 'perpendicular,' 'at right angles.' It has been usual to make an entirely unnecessary one: that the former should be used when the first line is looked upon as drawn from an external point to the second line (as in Prop. 12); the latter when the first line is looked upon as drawn from a point in the second (as in Prop. 11).

PROPOSITION 11. PROBLEM.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C the given point in it; it is required to draw from the point C a straight line at right angles to AB.



Take any point D in AC, and from CB cut off CE equal to CD. [I. 3.

On DE describe the equilateral △ DFE, [I. 1.
and join CF.

Then CF shall be at right angles to AB.

In the two $\triangle$s CFD, CFE

DC=CE, [CONST.

CF is common,

and DF=EF. [CONST.

$\therefore$ $\angle$ DCF= $\angle$ ECF [I. 8.

$\therefore$ they are right $\angle$s. [DEF. 10.

NOTES.

Compare this Proposition with Prop. 9.

If we followed the Syllabus in the so-called 'definition' of a plane $\angle$, we might speak of ACB as a *straight* $\angle$, and the problem is really equivalent to the following:—'To bisect a given straight $\angle$.' Hence the similarity between it and Prop. 9. As in Prop. 9 the demonstration would be just as easy if ACB were any isosceles $\triangle$ having AB for its base; and in Practical Geometry arcs with any equal radii long enough to ensure their cutting are described with centres A and B, and the points C and E on opposite sides of AB, when they cut, are joined, then the join of CE bisects AB. We do not join AC, CB, AE, EB because we are not concerned with demonstrating the correctness of our method; on the other hand Euclid does not draw in the other $\triangle$ AEB, because the construction of that or some such $\triangle$ is tacitly implied in the words 'bisect $\angle$ DFE.'

Ex. 7.—A and B are two given points, and C the mid-point of their join; P is any point on the line drawn through C $\perp$r to AB. Show that AP is equal to BP.

Ex. 8.—A and B are two given points, and C the mid-point of their join; P is any other point such that AP is equal to BP. Show that CP is $\perp$r to AB.

N.B.—Ex. 7 and 8 are particular enunciations of the following general theorems:—

1. If a point is on the perpendicular bisector of the join of two given points it is equidistant from them.

2. If a point is equidistant from two given points it is on the perpendicular bisector of their join.

The following enunciation is a concise statement *equivalent to* 1 *and* 2 :—

The locus of a point equidistant from two given points is the perpendicular bisector of their join.

The student should accustom himself as soon as possible to the use of the word **locus** and of the idea underlying it. A general definition will be found on page 107.

Ex. 9.—Find a point in a given straight line, XY, of unlimited length, equidistant from two given points, A, B.

(Refer to Note just given. *It is there pointed out that the locus of points equidistant from* A *and* B *is the perpendicular bisector of their join. Consequently the point required is the intersection of this perpendicular bisector with* XY.)

Ex. 10.—Find a point equidistant from two fixed points A and B, and also equidistant from two other fixed points C and D.

(*The point will be the intersection of two lines, each of which is a locus*).

Ex. 11.—**Find a point equidistant from three given points.**

Ex. 12.—**One diagonal of a kite bisects the other at right angles.**

Ex. 13.—**Each diagonal of a rhombus bisects the other at right angles.**

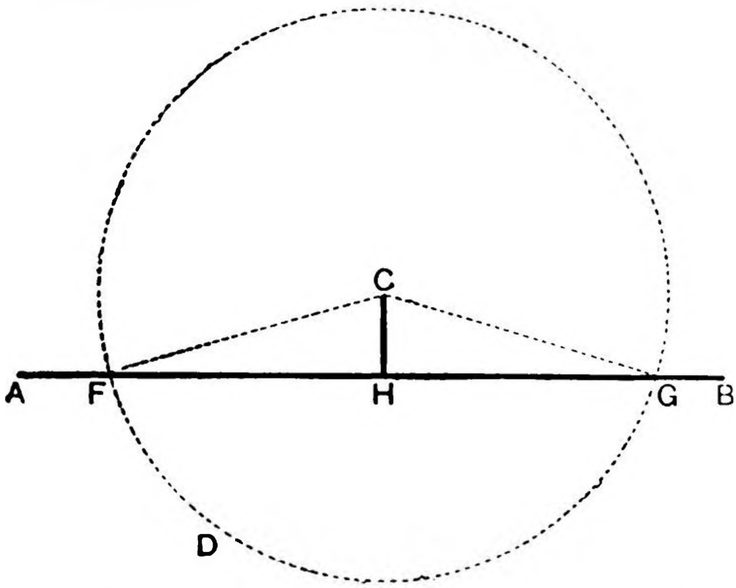
Ex. 14.—If *one* diagonal of a quadrilateral bisects the other at right angles the quadrilateral must be a kite.

Ex. 15.—If *each* diagonal of a quadrilateral bisects the other at right angles it must be a rhombus.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.

Let AB be the given straight line which may be produced to any length both ways, and let C be the given point without it; it is required to draw from the point C a straight line ⊥r to AB.



Take any point D on the other side of AB, and from the centre C at the distance CD describe ⊙ DGF, meeting AB at G and F

Bisect FG at H. [I. 10.

Join CH.

Then CH shall be ⊥r to AB.

Join CF, CG.

In the two △s HCF, HCG,

FH = HG, [Const.

HC is common,

and CF = CG. [Def. 15.

∴ ∠ CHF = ∠ CHG. [I. 8.

∴ CH is ⊥r to AB. [Def. 10.

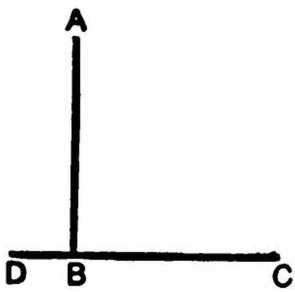
NOTE.

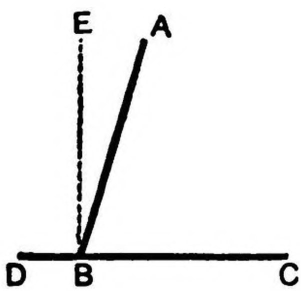
The student should go through the successive steps practically, and compare Euclid's process with that given in manuals on Practical Geometry. See note on Props. 9, 10, 11.

PROPOSITION 13. THEOREM.

The angles which one straight line makes with another straight line on one side of it are either two right angles or are together equal to two right angles.

Let the straight line AB make with the straight line CD on one side of it ∠s CBA, ABD, then ∠s CBA, ABD are either two right ∠s or together equal to two right ∠s.





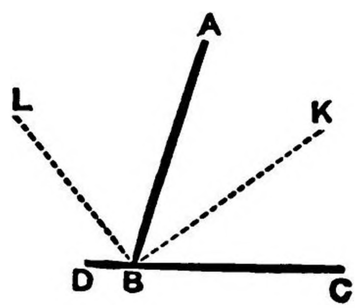
If ∠ CBA = ∠ ABD, then each of them is a right ∠. [DEF. 10.
If not, from point B draw BE ⊥r to CD. [I. 11.
Now ∠s CBA, ABE together = ∠ CBE;
∴ three ∠s CBA, ABE, EBD together = the two ∠s CBE, EBD. [AX. 2.
Again ∠ DBA = two ∠s DBE, EBA;
∴ ∠s DBA, ABC = three ∠s DBE, EBA, ABC. [AX. 2.
But these three have been shown equal to the two ∠s CBE, EBD,
and ∴ ∠s DBA, ABC = ∠s CBE, EBD, [AX. 1.
which are right ∠s. [CONST.

NOTES.

Angles like ABC, ABD, which are together equal to two right $\angle$s, are said to be **supplementary to each other**; and each is called the **supplement** of the other.

Ex. 16.—**If two adjacent supplementary angles, ABC, ABD, are bisected by straight lines, BK, BL, the angle KBL is a right angle.**

Angles like ABK, ABL, which are together equal to one right $\angle$, are said to be **complementary to each other**; and each is called the **complement** of the other.



Thus also in the second diagram of I. 13, the angles ABE, ABC are complementary to each other.

Ex. 16 (*a*).—When is an angle equal (1) to its *supplement*, (2) to its *complement*.

Ex. 16 (*b*).—Find an angle which is (1) one-third (2) one-seventh of its supplement. What fraction is it, in the latter case, of its complement?

Ex. 16 (*c*).—AD is drawn perpr. to the base BC of a $\triangle$ABC from the vertex A, and produced to D so that AD = DP. If P be joined to B and C, shew that $\triangle$s ABC, PBC are congruent.

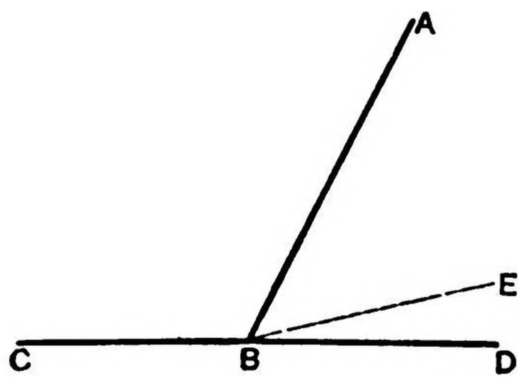
Ex. 16 (*d*).—Enunciate and prove the converse of Ex. 16 (*c*).

Ex. 16 (*e*).—A st. line RS meets a st. line TSV at S. On the other side of TSV is drawn SX. Show that if $\angle$VSX = $\angle$RST, then also $\angle$TSX = $\angle$RSV.

PROPOSITION 14. THEOREM.

If at a point in a straight line two other straight lines on opposite sides of it make adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the st. line AB, let the st. lines BC, BD on opposite sides of it make the adjacent ∠s ABC, ABD, together equal to two rt. ∠s;
then BC shall be cut in the same st. line as BD.



For if BD be not in the same st. line as BC, if possible let BE be in the same st. line with it.

Then ∠s ABC, ABE together = two rt. ∠s; [I. 13.
∴ ∠s ABC, ABE together = ∠s ABC, ABD, [Ax. 1.
∴ ∠ ABE = ∠ ABD, [Ax. 3.
which is impossible. [Ax. 9.
∴ BE is not in the same straight line with CB.

In the same way it may be shown that no other st. line through B except BD can be in the same st. line with CB,

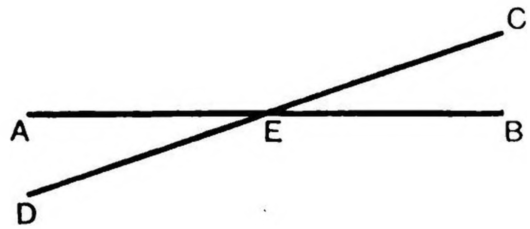
∴ BD is in the same st. line with CB.

PROPOSITION 15. THEOREM.

If two straight lines cut one another, the vertical or opposite angles are equal.

Let two st. lines AB, CD cut one another at point E,

then ∠AEC = ∠DEB, and ∠CEB = ∠AED.



The adjacent ∠s AEC, AED (made by AE with CD) = two rt. ∠s,
and the adjacent ∠s AED, DEB (made by DE with AB) = two rt. ∠s. [I. 13.

∴ ∠s AEC, AED = ∠s AED, DEB; [Ax. 1.

∴ ∠AEC = ∠DEB. [Ax. 3.

In the same way it may be shown that ∠CEB = ∠AED.

COROLLARY I.—**If two straight lines cut one another, the angles they make at the point where they cut are together equal to four right angles.**

COROLLARY II.—**All the angles made by any number of straight lines meeting at one point are together equal to four right angles.**

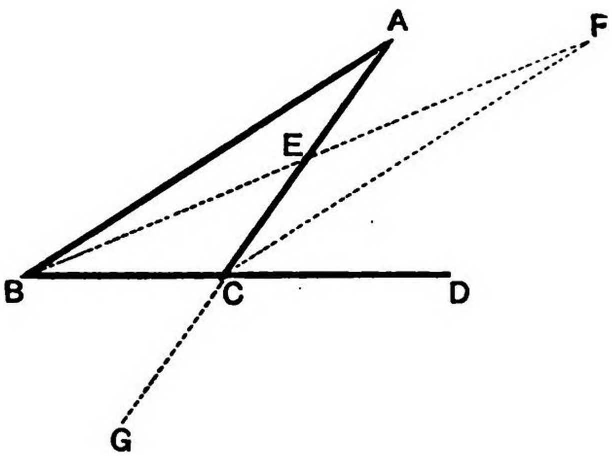
Ex. 17.—The four st. lines EF, EG, EH, EK drawn from E, bisecting the ∠s AEC, CEB, BED, DEA, form two st. lines FH, GK at rt. ∠s to each other. (*See* Ex. 16.)

Ex. 18.—If four st. lines EA, EC, EB, ED be drawn from a point E, making the ∠s AEC, CEB respectively equal to their opposite ∠s BED, DEA, they form two st. lines AB, CD.

PROPOSITION 16. THEOREM.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior and opposite angles.

Let the side BC of the △ ABC be produced to D, the ext. ∠ ACD shall be greater than either of the int. and opp. ∠s BAC, ABC.



Bisect AC in E. [I. 10.

Join BE and produce it to F, making EF equal to EB, [I. 3.
and join FC.

Now in △s AEB and CEF

AE, EB=CE, EF, respectively, [CONST.

and ∠ AEB= ∠ CEF. [I. 15.

∴ ∠ BAE= ∠ ECF. [I. 4.

But ∠ ACD> ∠ ECF. [AX. 9.

∴ ∠ ACD> ∠ BAC.

In the same manner, if BC be bisected and the side AC produced to G, it may be shown that ∠ BCG> ∠ ABC.

But ∠ BCG= ∠ ACD. [I. 15.

∴ ∠ ACD> ∠ ABC.

NOTE.

Students are exceedingly apt to make a slip in trying to do the second part of this proposition, and say 'similarly it may be shown that the ∠ ACD is also greater than the ∠ ABC,' which is not correct.

A line like BE *drawn from an angle of a triangle to the mid-point of the opposite side is called* **a median line,** *or simply* **a median** *of the triangle.*

Ex. 19.—Enunciate the converse of Prop. 16.

Ex. 20.—Two of the medians of an isosceles triangle are equal.

Ex. 21.—The three medians of an equilateral triangle are equal.

Ex. 21 (*a*).—In the fig. of I. 16 shew that the three ∠s of △ FBC are together equal to the three ∠s of △ ABC.
(Lobatschewsky makes use of this theorem in his *Theory of Parallels.*)

Ex. 21 (*b*).—In the fig. of I. 16 shew that △ FBC = △ ABC.

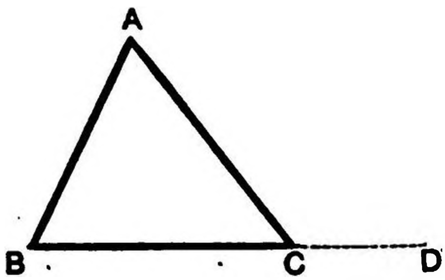
Ex. 21 (*c*).—If in a △ ABC, BC = CA, and the ext. ∠ ACD were bisected, then the bisecting st. line and the side AB could not meet, however far they were produced. *Prove indirectly.*
Enunciate this theorem generally.

Ex. 21 (*d*).—If in the fig. of I. 5 FG were joined, the st. lines FG, BC would never meet, however far they were produced. *Prove indirectly.*
Enunciate this theorem generally.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be a △, any two of its ∠s are together less than two rt. ∠s.



Produce BC to D.

Then ext. ∠ACD > int. and opp. ∠ABC. [I. 16.

the ∠s ACD, ACB together > ∠s ACB, ABC. [Ax. 4.

But ∠s ACD, ACB together = two rt. ∠s. [I. 13.

∴ ∠s ABC, ACB together < two rt. ∠s.

In the same manner it may be shown that ∠s BAC, ACB, and also ∠s CAB, ABC together < two rt. ∠s.

Ex. 22.—Enunciate the converse of this Theorem.

Ex. 23.—Not more than two straight lines equal to one another can be drawn from a given point to a given straight line. (Prove *indirectly.*)

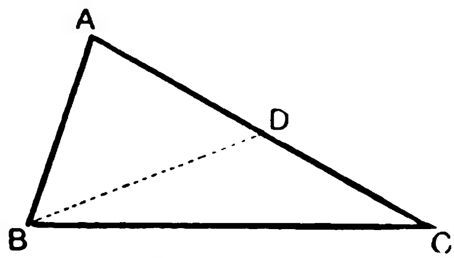
Ex. 23 (*a*).—Each of the ∠s at the base of an isosceles △ is acute.

Ex. 23 (*b*).—No △ can have more than one right or obtuse angle.

PROPOSITION 18. THEOREM.

The greater side of every triangle has the greater angle opposite to it.

Let the side AC of the $\triangle$ ABC be greater than the side AB, then $\angle$ ABC > $\angle$ ACB.



From AC cut off AD equal to AB, [I. 3.
and join BD;
$\because$ AB=AD,
$\therefore$ $\angle$ ADB= $\angle$ ABD; [I. 5.
but $\angle$ ABC> $\angle$ ABD, [AX. 9.
$\therefore$ $\angle$ ABC> $\angle$ ADB.
But ext. $\angle$ ADB> int. and opp. $\angle$ ACB. [I. 16.

$\therefore$ $\angle$ ABC> $\angle$ ACB.

NOTE.

It should be carefully remembered that this proposition is merely equivalent to the following:—

If two sides of a triangle are unequal, the angle subtended by the greater is greater than the angle subtended by the less.

The proposition includes the following:—

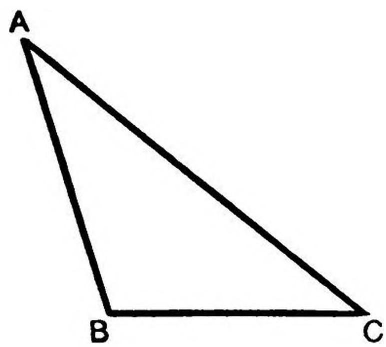
If two sides of a triangle are unequal, the angles which they subtend are unequal.

This last is called the **obverse** of Proposition 5, and is an immediate inference from Proposition 6.

PROPOSITION 19. THEOREM.

The greater angle of every triangle is subtended by the greater side or has the greater side opposite to it.

Let ∠ABC of △ABC be greater than ∠ACB, then the side AC > the side AB.



For AC either = AB or < AB or > AB.

If AC = AB,

∠ABC = ∠ACB, [I. 5.

which is contrary to hypothesis.

If AC < AB,

∠ABC < ∠ACB [I. 18

which is contrary to hypothesis.

∴ AC > AB.

Ex. 23 (*c*).—Prove I. 18 by bisecting ∠BAC by AH meeting BC in H and joining HD.

Ex. 23 (*d*).—Prove I. 6 by I. 19. Is this a fair proof? (*Yes, for* I. 6 *has not yet been used.*)

NOTE.

This is the converse of Prop. 18.

We have here a second instance of the use of the indirect method for proving a converse (see Prop. 6).

It should be carefully remembered that this proposition is merely equivalent to the following :—

'If two angles of a triangle are unequal, the side which subtends the greater is greater than the side subtending the less.'

The proposition includes the following :—

'If two angles of a triangle are unequal, the sides which subtend them are unequal.'

This is the **obverse** of Prop. 6, and is an immediate inference from Prop. 5.

Ex. 24.—**Of all straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest; and of the others those which make equal angles with the perpendicular are equal; and that which makes a greater angle with the perpendicular is greater than that which makes a less angle.**

Ex. 24 (*a*).—In the fig. of I. 16 if AB > BC then $\angle$ABE < $\angle$CBE.

Ex. 24 (*b*).—In $\triangle$ABC if AD, bisecting $\angle$BAC meets BC in D, then AB > BD and AC > CD. Also if AB > AC, then BD > DC.

Ex. 24 (*c*).—A st. line PQR is drawn cutting the equal sides AB, AC of an isos. $\triangle$ABC in P, Q and BC produced in R. Shew that $\angle$APQ > $\angle$CQR and hence that AQ > AP. State and prove the converse.

Ex. 24 (*d*).—A st. line PQR is drawn cutting the equal sides AB, AC produced in P, Q and BC produced in R. Shew that AP > AQ. State and prove the converse.

Ex. 24 (*e*).—If in the fig. of I. 19, BC be produced through C to any pt. D, and AD be joined, then AD>AB.

Ex. 24 (*f*).—If the base BC of an isosceles $\triangle$ABC be produced to any pt. D, and AD be joined, then AD is greater than either of the equal sides AB, AC.

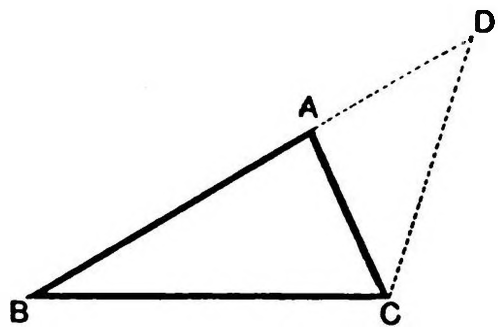
Ex. 24 (*g*).—If a pt. D in the base BC of an isosceles $\triangle$ABC, and AD be joined, then AD is less than either of the equal sides AB, AC.

Show also that if D divides the base BC unequally, then AD divides the vertical angle BAC unequally and conversely.

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third.

Let ABC be a △, any two sides of it are together greater than the third side.



Produce BA to D, making AD equal to AC; [I. 3.
join CD.
Then ∵ AD = AC, [CONST.
∴ ∠ADC = ∠ACD. [I. 5.
But ∠BCD > ∠ACD, [AX. 9.
∴ ∠BCD > ∠ADC.
∴ side BD of △ BCD opp. to ∠BCD > side BC in same △ opp. ∠ ADC. [I. 19.
But BD = BA and AC, [CONST.

∴ BA, AC > BC.

In the same manner it may be shown that AB, BC > AC, and that BC, CA > BA.

Ex. 25.—**Any side of a triangle is greater than the difference of the other two sides.**

Prove this (1) independently, (2) as a deduction from Prop. 20.

Ex. 26.—Any three sides of a quadrilateral are together greater than the fourth.

Ex. 27.—Any two sides of a triangle are together greater than twice the median line which bisects the base (see figure of Prop. 16).

Ex. 28.—The sum of the distances of any point from the three vertices of a triangle is greater than half the sum of the sides.

Ex. 29.—Enunciate and prove a similar theorem as regards a quadrilateral.

Can it be extended to other polygons (figures with more sides than three)?

The four sides of a quadrilateral are together greater than the diagonals.

Are the five sides of a pentagon (five-sided figure) greater than its five diagonals?

Ex. 30.—Two triangles are constructed, each by joining the alternate vertices of a hexagon (six-sided figure).

Show that the perimeter of the hexagon is greater than half the sum of the perimeters of the triangles.

Ex. 31.—ABCDEF is a hexagon. Show that its perimeter is greater than two-thirds of the sum of the three diagonals AD, BE, CF.

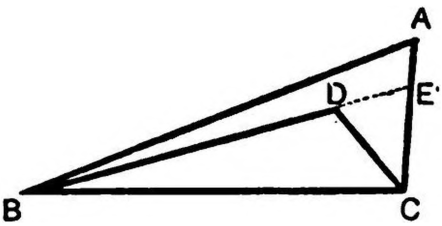
Ex. 31 (*a*).—The side BC of a $\triangle$ ABC is bisected in D. Show that if BA=AC each of these sides is greater than BD, and that in any case one of them must be greater than BD.

Ex. 31 (*b*).—If one side AB of a $\triangle$ ABC is less than half BC, what can we infer about CA?

PROPOSITION 21. THEOREM.

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

From the ends B, C of the side BC of the △ ABC let the two st. lines BD, CD be drawn to a point D within the △; BD, DC shall be less than BA, AC but ∠ BDC shall be greater than ∠ BAC.



Produce BD to meet AC in E.

Now BA, AE > BE, [I. 20.

∴ BA, AE, EC > BE, EC.

i.e. BA, AC > BE, EC.

Again DE, EC > DC, [I. 20.

∴ BD, DE, EC > BD, DC.

i.e. BE, EC > BD, DC.

But BA, AC > BE, EC, [DEMON.

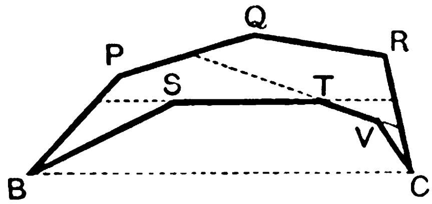
∴ BA, AC > BD, DC.

Again ext. ∠ BDC > int. and opp. ∠ BEC. [I. 16.

Also ext. ∠ BEC > int. and opp. ∠ BAC. [I. 16.

∴ ∠ BDC > ∠ BAC.

Ex. 32.—Straight lines, BP, PQ, QR, RC, are drawn forming a **rectilineal path,** BPQRC, from B to C but not crossing BC. A second rectilineal path, BSTVC, is drawn from B to C lying entirely between the first path and BC, *and such that if any of the straight lines* BS, ST, TV, VC *which form it be produced either way, the produced parts will lie without the figure* BSTVC. Show that *the outside path is the longer.*



(A generalisation of the first part of Prop. 21, to be proved in the same way.)

Examine the effect of omitting the words in italics from the hypothesis.

Ex. 33.—O is any point within the triangle ABC, show that OA, OB, OC are together less than AB, BC, CA together.

Ex. 33 (*a*).—O is any pt. within an equil. △ ABC. Shew (1) that any one of the three st. lines OA, OB, OC is less than a side of the △ ABC; (2) that any two of them are together greater than the third.

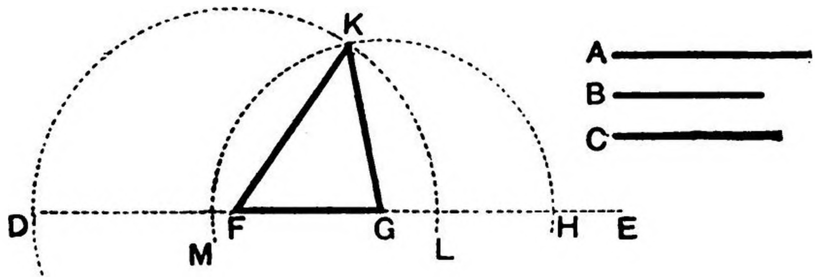
Ex. 33 (*b*).—O is any pt. within an equil. △ ABC. Equil. △s APO AQO are described on AO, APO being on the same side of AO as B is. Shew that each of the △s OPB, OQC has its sides equal to OA, OB OC.

Postpone this Ex. to I. 32.

PROPOSITION 22. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, any two of which are together greater than the third.

Let A, B, C be the three given straight lines of which any two whatever > the third; it is required to make a △ of which the sides shall be equal to A, B, C, each to each.



Take a straight line DE, terminated at the point D, but unlimited towards E,

cut off DF equal to A, FG equal to B, and GH equal to C. [I. 3.

From centre F, at distance FD, describe ⊙ DKL,

from centre G, at distance GH, describe ⊙ HKM cutting ⊙ DKL in K.

Join KF, KG. The △ KFG shall have KF equal to A, FG to B, and GK to C.

Now radius FK=radius FD; [DEF. 15.

∴ FK=A. [AX. 1.

Again radius GK=radius GH;

∴ GK=C, [AX. 1.

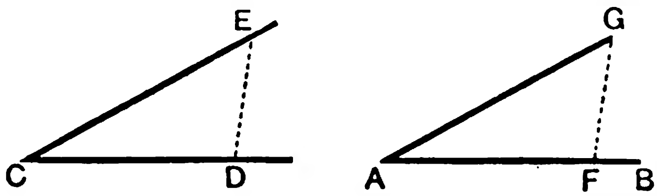
and FG=B;

∴ the three straight lines KF, FG, GK=A, B, C.

PROPOSITION 23. PROBLEM.

At a given point in a given straight line to make an angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the point in it, and DCE the given rectilineal angle; at the point A in the given straight line AB, it is required to make an angle equal to $\angle$ DCE.



In CD, CE take any points D and E and join DE.

From AB cut off AF equal to CD and make $\triangle$ AFG, the sides AF, AG, FG of which = the three straight lines CD, CE, ED respectively; [I. 22.

then $\angle$ FAG = $\angle$ DCE. [I. 8.

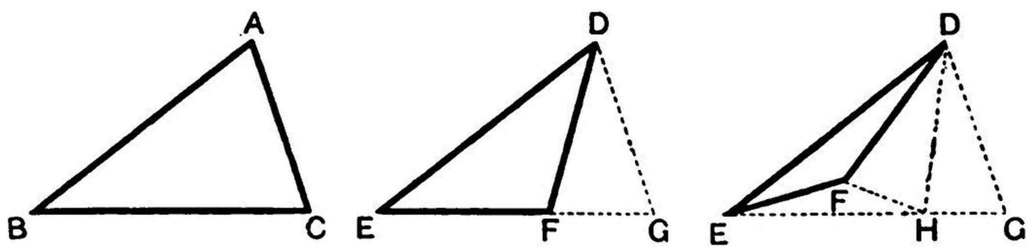
NOTE.

In Practical Geometry it is usual to make the two lines CD, CE *equal* (by describing an arc with centre C and any radius); and then to make a triangle FAG congruent with DCE (by describing arcs with centres A and F, and radii CD and DE, respectively).

PROPOSITION 24. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two $\triangle$s having the sides AB, AC equal to the sides DE, DF, each to each, but $\angle$ BAC greater than $\angle$ EDF,
then the base BC shall be greater than the base EF.



At the point D in the straight line DE make $\angle$ EDG equal to $\angle$ BAC, [I. 23.
and make DG equal to AC or DF, [I. 3.
and join EG.

If EG passes through F we have at once EG>EF.

If EG does not pass through F, bisect $\angle$ FDG by the straight line DH, meeting EG in H; join HF. [I. 9.

Then in $\triangle$s FDH, GDH
FD, DH=GD, DH,
and $\angle$ FDH=$\angle$ GDH;
$\therefore$ HF=HG;
$\therefore$ EH, HF=EG. [AX. 2.

But EH, HF>EF; [I. 20.

∴ EG>EF.

Now in either case in the two △s BAC, EDG

BA, AC=ED, DG,

and ∠ BAC=∠ EDG; [CONST.

∴ BC=EG.

But EG>EF;

∴ BC>EF.

NOTE.

An obverse of Prop. 4 is included in this Proposition. See p. 37.

Ex. 33 (*c*).—Prove I. 24 by a construction similar to the above but having the angle EDG made on the opposite side of ED to the angle EDF.

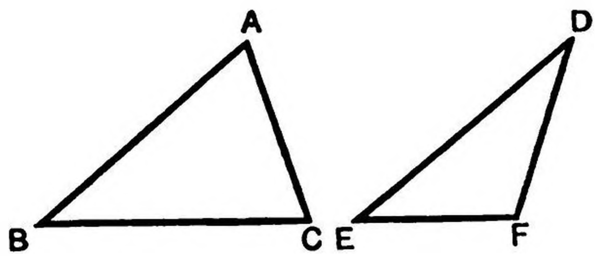
Ex. 33 (*d*).—If in the fig. of I. 5, CG were made equal to AC, show that BG>AC.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let ABC and DEF be two $\triangle$s having the two sides AB, AC equal to the two sides DE, DF respectively, but the base BC greater than the base EF,

then $\angle$ BAC > $\angle$ EDF.



For $\angle$ BAC either = $\angle$ EDF or < $\angle$ EDF or > $\angle$ EDF.

If $\angle$ BAC = $\angle$ EDF,

BC = EF, [I. 4.

which is contrary to the hypothesis;

if $\angle$ BAC < $\angle$ EDF,

BC < EF, [I. 24.

which is also contrary to the hypothesis;

$\therefore$ $\angle$ BAC > $\angle$ EDF.

NOTE.

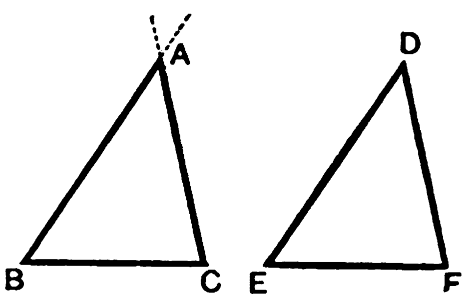
This Proposition is the converse of Prop. 24.
It includes the obverse of Prop. 8.

Ex. 34.—Write down the obverse of Prop. 8.
From what Proposition is it an immediate inference?

PROPOSITION 26. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let the two △s ABC, DEF have ∠ B equal to ∠ E, and ∠ C equal to ∠ F; and first let those sides be equal which are adjacent to the equal angles in the two △s, namely BC to EF; then shall AB, AC be equal to DE, DF, each to each, and the third ∠ A be equal to the third ∠ D.



For if △ DEF be applied to the △ ABC so that the point E may be on the point B, and the straight line EF on the straight line BC, the point F will coincide with the point C ∵ EF=BC; and the straight line ED must fall on the straight line BA, ∵ ∠ E= ∠ B, and ∴ the point D must lie somewhere on the straight line from B through the point A. [HYP.

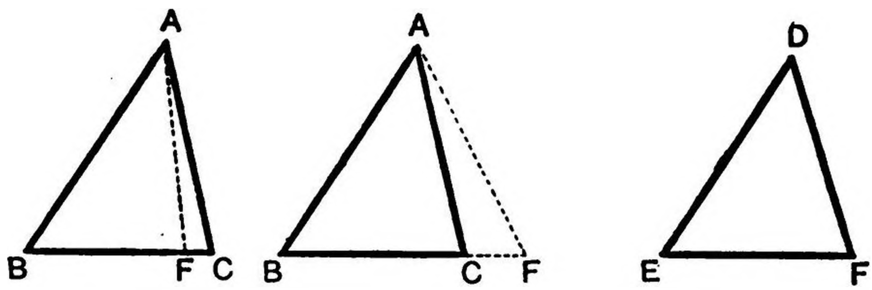
Similarly the pt. D must lie somewhere on the straight line from C through A, ∵ ∠ F= ∠ C; [HYP.

∴ it must coincide with A,

and ∴ the whole △ DEF coincides with the whole △ ABC and=it,

and AB = DE, and AC = DF,
and ∠ BAC = ∠ EDF.

Next let the two △s ABC, DEF have the ∠s B and C respectively equal to the ∠s E and F, as in the first case, but let them have sides AB, DE, opposite the equal ∠s C and F, equal; then shall AC, CB be equal to DF, FE, each to each, and the third ∠ A to the third ∠ D.



For if △ DEF be applied to △ ABC so that the point D is on the point A and the straight line DE on the straight line AB, the point E will coincide with the point B, ∵ DE = AB;
and EF must fall on the straight line BC, ∵ ∠ E = ∠ B;
and ∴ the point F must fall somewhere on the straight line from B through C.

If it did not coincide with C we should have two angles AFB, ACB equal to each other, one of which was an exterior angle of the △ AFC, and the other an interior angle opposite to it, which is impossible; [I. 16.
∴ F coincides with C;
and ∴ the whole △ DEF coincides with the whole △ ABC,
and has DF, FE coinciding respectively with AC, CB,
and ∠ D with ∠ A.

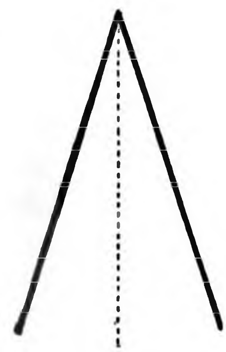
Ex. 35.—Write down the obverse of Prop. 4.

From what proposition is it the immediate inference?

Ex. 36.—The two perpendiculars let fall from any point on the line bisecting an angle to the two straight lines which contain it are equal.

1. The converse of this Theorem is also true.

2. By the *distance* of a point from a straight line we mean the length of the perpendicular from the point to the straight line. Hence :—

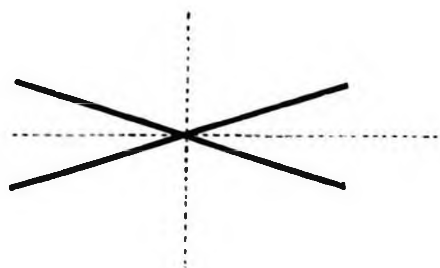


The locus of a point equidistant from two straight lines drawn from the same point is the bisector of the angle contained by them.

For the same use of the word locus see Note on Ex. 8.

The following proposition is slightly more general :—

The locus of a point equidistant from two given intersecting straight lines is the pair of lines at right angles to one another which bisect the angles made by the given lines. (Syllabus.) (See Ex. 16 and Ex. 18.)



Ex. 37.—Find a point on the base of a triangle equidistant from its two sides. (The point will be the intersection of a locus with the base.)

Ex. 38.—**Find a point within a given triangle equidistant from the three sides.** (*The point required is at the intersection of two loci.*)

Ex. 39.—If any straight line be drawn through the mid-point of the join of two fixed points, it is equidistant from them.

Ex. 39 (*a*).—Prove I. 25 directly thus:—From BC cut off BG = EF and on BG describe a $\triangle$ with its sides BH, HG = DE, DF. Let HG produced cut AC in I. Join AH and prove $\angle$ IAH > $\angle$ IAH. (*Menelaus.*)

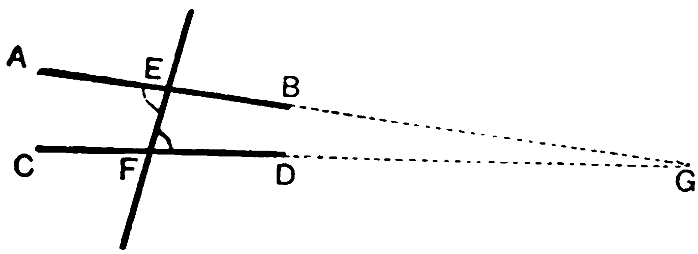
Def.—Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

The symbol for 'parallel' is ||.

PROPOSITION 27. Theorem.

If a straight line falling on two other straight lines makes the alternate angles equal to one another, the two straight lines shall be parallel.

Let the straight line EF, falling on AB, CD, make the alternate angles AEF, EFD equal; AB shall be parallel to CD.



For if not, AB, CD, if produced, will meet either through B, D or A, C.

Let them be produced through B, D, and if possible let them meet in G;

then GEF is a $\triangle$,

and $\therefore$ ext. $\angle$ AEF $>$ int. and opp. $\angle$ EFG; [I. 16.

but $\angle$ AEF $=$ $\angle$ EFD; [Hyp.

$\therefore$ AB and CD being produced do not meet through B, D.

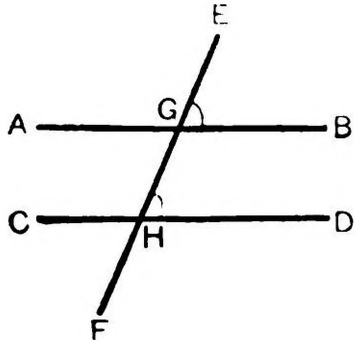
In like manner it may be shown that they do not meet if produced through A, C;

$\therefore$ AB is || to CD. [Def. 35.

PROPOSITION 28. THEOREM.

If a straight line falling on two other straight lines makes the exterior angle equal to the interior and opposite angle on the same side of the line, or makes the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel.

Let the st. line EF falling on the two st. lines AB, CD make ext. ∠ EGB equal to int. and opp. ∠ GHD on the same side, or make the two int. ∠s on the same side, BGH, GHD, together equal to two rt. ∠s; then AB shall be || to CD.



∵ ∠ EGB = ∠ GHD,	[HYP.
and ∠ EGB = ∠ AGH ;	[I. 15.
∴ ∠ AGH = ∠ GHD ;	[AX. 1.
∴ AB is || to CD.	[I. 27.
Again ∵ ∠s BGH, GHD together = two rt. ∠s,	[HYP.
and ∠s AGH, BGH also = two rt. ∠s ;	[I. 13.
∴ ∠s AGH, BGH = ∠s BGH, GHD.	[AX. 1.
∴ ∠ AGH = ∠ GHD.	[AX. 3.
∴ AB is || to CD.	[I. 27.

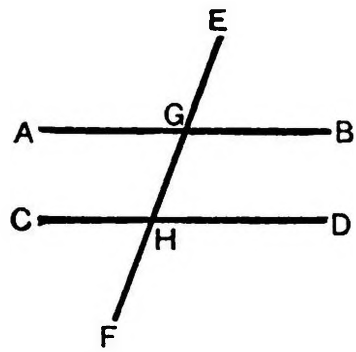
Ex. 40.—Prove this proposition independently of Prop. 27.

Ax. 12.—**If a straight line meet two other straight lines so as to make the two interior angles on the same side of it together less than two right angles, these two straight lines being continually produced shall at length meet on that side on which are the angles which are together less than two right angles.**

PROPOSITION 29. THEOREM.

If a straight line fall on two parallel straight lines, it makes the alternate angles equal, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the st. line EF fall on the two || st. lines AB, CD, the alternate ∠s AGH, GHD shall be equal, and ext. ∠ EGB shall be equal to int. and opp. ∠ on the same side, GHD, and the two int. ∠s on the same side, BGH, GHD, shall together=two rt. ∠s.



For if ∠ AGH be not equal to ∠ GHD, one of them must be the greater: let ∠ AGH be the greater.

Then ∵ ∠ AGH > ∠ GHD,
∴ ∠s AGH, BGH > ∠s BGH, GHD.

But ∠s AGH, BGH = 2 rt. ∠s; [I. 13.
∴ ∠s BGH, GHD < 2 rt. ∠s.
∴ AB and CD will meet if produced. [AX. 12.

But they never meet, since they are ||; [HYP.
∴ ∠ AGH is not unequal to ∠ GHD, *i.e.* is equal to it.

But ∠ AGH= ∠ EGB; [I. 15.
∴ ∠ EGB= ∠ GHD; [AX. 1.
∴ ∠s EGB, BGH= ∠s BGH, GHD. [AX. 2.

But ∠s EGB, BGH=2 rt. ∠s; [I. 13.
∴ ∠s BGH, GHD=2 rt. ∠s. [AX. 1.

NOTES.

Axiom 12, on which the proof of I. 29 is based, is not probably self-evident to anybody who has not thought somewhat deeply on the nature of parallels. It is the converse of I. 17. The student may perhaps find less difficulty in admitting as an evident truth the statement that '**Two intersecting straight lines cannot both be parallel to the same straight line**,' or '**Through a given point only one parallel can be drawn to a given straight line**,' from either of which Euclid's so-called '12th Axiom' can be deduced.

On the nature of this assumption see Henrici (Congruent Figures, pp. 65-69).

The three parts of Prop. 29 are converses of Prop. 27 and of the two parts of Prop. 28 respectively.

Ex. 41.—If any st. line be ∥ to the join of two points, it is equidistant from those points. (Join one of the given points with the foot of the ⊥r of the other, and use 29 and 26.)

Compare Ex. 39.

Ex. 42.—If a st. line be equidistant from two given points, it either bisects their join, or is ∥ to it.

Ex. 43.—If any number of st. lines be drawn at rt. ∠s to a given st. line, they are all ∥ to one another.

A number of parallel straight lines is sometimes spoken of as a **pencil.**

A number of straight lines all passing through a given point is also spoken of as a **pencil.**

All the straight lines which satisfy the same geometrical condition are spoken of as a **set.** See Henrici (*Congruent Figures*, pp. 137-146.)

Thus Ex. 39, 41, 42 might be enunciated thus :—

The set of lines equidistant from two given points form two pencils, one parallel to their join and the other bisecting it.

Ex. 44.—**The set of lines making equal angles with two given intersecting straight lines form two parallel pencils.**

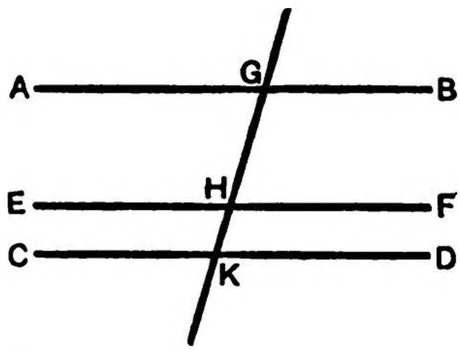
Ex. 45.—**The set of lines making a given acute or obtuse angle with a given straight line form two sets of parallels.**

What alteration has to be made in the above enunciation if we substitute **right** for **acute** or **obtuse**?

PROPOSITION 30. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.

Let AB, CD be each of them || to EF; then AB shall be || to CD.



Let st. line GHK cut AB, EF, CD in G, H, K.

Then ∵ GHK cuts the ||s AB, EF;

∴ ∠ AGH = ∠ GHF, [I. 29.

and ∵ GHK cuts the ||s EF, CD;

∴ ∠ GHF = ∠ GKD; [I. 29.

∴ ∠ AGH = ∠ GKD; [Ax. 1.

∴ AB is || to CD. [I. 27.

NOTE.

This proposition is an immediate inference from Playfair's Axiom, 'Two intersecting straight lines, etc.,' see page 55.

Ex. 46.—What is the converse of this proposition?

Ex. 47 (i.).—Through a given point draw as many st. lines as you can making a given ∠ with a given st. line.

The required straight lines will be those particular members of a certain set *which pass through the given point.* (See Ex. 45.)

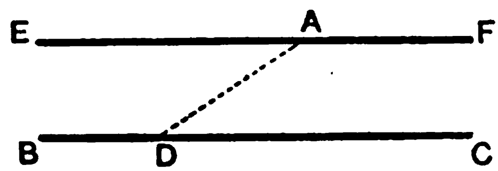
(ii.).—Through a given point draw as many st. lines as you can, making equal ∠s with two given intersecting st. lines.

The required straight lines are those particular members of a certain set which pass through the given point. (See Ex. 44.)

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point and BC the given st. line; it is required to draw through A a st. line || to BC.



In BC take any point D and join AD;

At the point A in the st. line AD make ∠ DAE equal to alt. ∠ ADC, [I. 23.

and produce EA to F;

EF shall be parallel to BC.

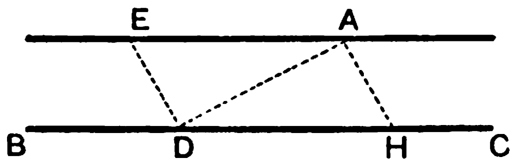
∵ st. line AD, meeting the two st. lines BC, EF, makes the ∠ EAD = alt. ∠ ADC, [CONST.

∴ EF is || to BC. [I. 27.

Ex. 48.—Two congruent triangles can be placed so as to have one side common, and the other equal sides parallel each to each.

NOTE.

A method given in works on Practical Geometry for drawing through a given point a straight line parallel to a given straight line is worth noticing on account of its simplicity and its connection with the above theorem.



Let A be the given point, and D, H *any two points* in the given straight line. With centres A and D, and radii equal to DH, AH respectively, describe arcs cutting in a point E, on the opposite side of AD to H. Then AE shall be the required straight line. If we join ED, DA, AH we can easily show that the triangles HAD, ADE are congruent (I. 8), and have sides AE, DE parallel respectively to BC, AH (I. 27).

When a straight line is drawn *from* a point A parallel to given st. line BC, it may be drawn in one of two opposite directions. Supposing AB to be joined it may be drawn (1) on the same side of AB as BC is (like AF in the fig. of I. 31); (2) on the opposite side of AB to BC like AE.

It is often convenient to distinguish between these two lines, and the word 'sense' is used for the purpose. Thus AF is said to be 'parallel to BC *and in the same sense*,' while AE is said to be 'parallel to BC *and in the opposite sense.*'

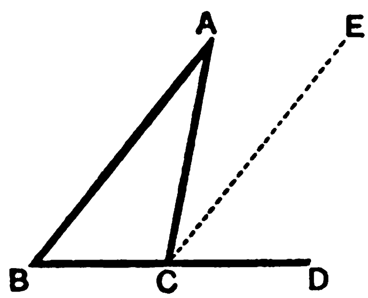
Ex. 48 (*a*).—In the diagram of the note compare the senses of (i) AE, DH; (ii) AE, CH; (iii) EA, DB.

PROPOSITION 32. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a △ having one side BC produced to D;

then ext. ∠ ACD = two int. and opp. ∠ s CAB, ABC, and the three int. ∠ s ABC, BAC, ACB together = two rt. ∠ s.



Through C draw CE parallel to AB. [I. 31.

Then ∠ BAC = alt. ∠ ACE, [I. 29.

and ext. ∠ ECD = int. and opp. ∠ ABC; [I. 29.

∴ whole ext. ∠ ACD = int. and opp. ∠ s CAB, ABC; [AX. 2.

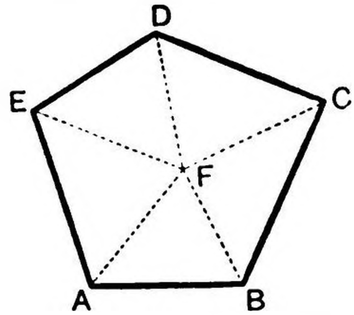
∴ ∠ s ACD, ACB = ∠ s CAB, ABC, BCA. [AX. 2.

But ∠ s ACD, ACB together = two rt. ∠ s; [I. 13.

∴ ∠ s CBA, BAC, BCA together = two rt. ∠ s. [AX. 1.

N.B.—Since the time of Barrow (1655) it has been customary to add the two following corollaries, which are now generally regarded as book work though not given by Euclid.

Corollary I.—**All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.**



For any rectl. figure, ABCDE, can be divided into as many △s as the figure has sides by drawing st. lines from a point F within the figure to each of its ∠s.

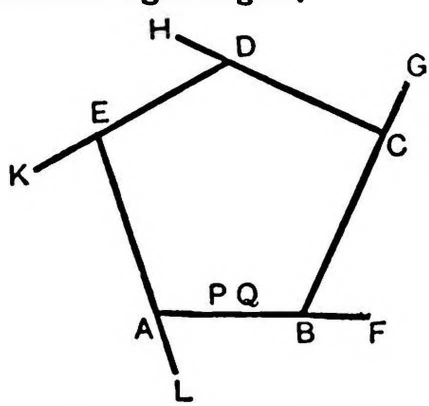
And by the preceding proposition all the ∠s of these △s = twice as many rt. ∠s as there are △s, *i.e.* as the figure has sides.

But the same ∠s = int. ∠s of the figure, together with the ∠s at the pt. F.

∴ all the int. ∠s of the figure, together with the ∠s at F, = twice as many rt. ∠s as the figure has sides.

∴ all the int. ∠s of the figure, together with four rt. ∠s, = twice as many rt. ∠s as the figure has sides. [I. 15. Cor. II.

Corollary II.—**All the exterior angles of any rectilineal figure are together equal to four right angles.**



∵ every int. ∠ABC, together with its adjacent ext. ∠CBF, = two rt. ∠s. [I. 13.

∴ all the int. ∠s, together with all the ext. ∠s = twice as many rt. ∠s as the figure has sides.

∴ all the int. ∠s, together with all the ext. ∠s = all the int. ∠s, together with four rt. ∠s. [I. 32. Cor. I.

∴ all the ext. ∠s together = four rt. ∠s. [Ax. 3.

NOTES.

Suppose a small st. line PQ to slide successively along the sides AB, BC, CD, DE, EA, starting with the end P at A, and sliding along AF until P comes to B, then turning about B until Q comes on BC, then sliding along BC until P comes to C, then turning about C until Q comes on CD, and so on round the figure, continuing the process until P has reached A again and Q has fallen on AB: PQ will thus have come back to its original position, and the student will probably admit it as axiomatic that it must have turned through four rt. ∠s altogether. But the ∠s it has really turned through are the ext. ∠s of the figure. Hence the truth of Cor. II.

A rectilineal figure, which is **equilateral** and **equiangular**, is called **regular**, *e.g.* an equilateral △ is a regular figure.

Ex. 49.—If a small st. line slide along the outside rectilineal path BPQRC, turning in the manner above described at the angles, show that the whole amount of turning will be greater than if it were to slide in the same manner along the inside path BSTVC (see Ex. 32).

Ex. 50.—The angle of an equilateral triangle has always the same magnitude.

Ex. 51.—The interior angle of any regular figure with a given number of sides has always the same magnitude.

Ex. 52.—The interior angle of a regular hexagon is equal to the exterior angle of an equilateral triangle, and *vice versa.*

Ex. 53.—Trisect a right angle.

Ex. 54.—The straight line parallel to the base of an isosceles triangle through the vertex bisects the exterior angle.

Ex. 55.—Enunciate and prove the converse of the preceding Exercise.

Ex. 56.—If a straight line be drawn parallel to the base of an isosceles triangle, it forms in general another isosceles triangle with the sides.

Ex. 57.—In the figure of I. 5 the join of FG is parallel to BC.

Ex. 58.—Show that six equal regular triangles can be arranged round one common vertex, so as to make a regular hexagon.

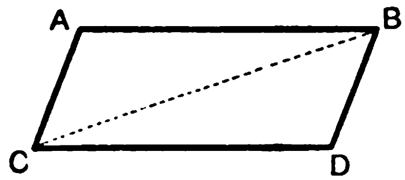
Ex. 59.—Show that four congruent right-angled isosceles triangles can be arranged round one common vertex so as to make a square (regular quadrilateral).

Ex. 60.—Enunciate and prove a theorem similar to Ex. 58, 59, with respect to a regular *octagon* (eight-sided figure). What would be the magnitude of an *exterior* angle of such a figure? (For finding *exterior* angles of regular figures use Cor. II.)

PROPOSITION 33. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

Let AB and CD be equal and parallel st. lines, and let them be joined towards the same parts by the st. lines AC and BD. AC and BD shall be equal and parallel.



Join BC,

$\because$ AB is $\parallel$ to CD ; [HYP.

$\therefore$ $\angle$ ABC = alt. $\angle$ BCD ; [I. 29.

and also in $\triangle$s ABC, DCB

AB, BC = CD, CB,

$\therefore$ AC = BD, [I. 4.

and $\angle$ ACB = $\angle$ CBD, [I. 4.

$\therefore$ AC is $\parallel$ to BD. [I. 27.

Ex. 61.—If we omitted the words 'towards the same parts' from the **hypothesis of I. 33,** what modification should **we** have to make in the **conclusion ?**

Ex. 61 (*a*).—EF, GH are two equal and parallel st. lines. Show that EG, HF are parallel to, or bisect, one another according as EF, GH are in the same sense or in opposite senses. (See p. 59.)

Def.—A parallelogram is a quadrilateral which has its opposite sides parallel.

The diagonal of a quadrilateral is the straight line joining two of its opposite angles.

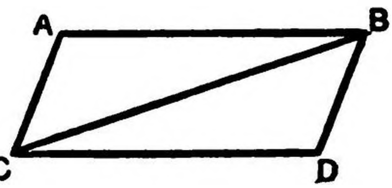
Diagonals of a parallelogram are sometimes called **diameters.**

The symbol for parallelogram is ||gm.

PROPOSITION 34. Theorem.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram, *i.e.* divides it into two equal parts.

Let ACDB be a ||gm of which BC is a diameter; the opposite sides and ∠s of the figure shall be equal to one another, and the diameter BC shall bisect it.



∠ABC=alt. ∠BCD, [I. 29.

and ∠ACB=alt. ∠CBD, [I. 29.

and BC is common to the two △s ACB, DCB, and adjacent to their equal ∠s;

∴ AB=CD,
AC=BD,
∠BAC=∠BDC,
and △ABC=△BCD
[Demonstration of I. 26.

(*i.e.* the ||gm is bisected by BC).

Similarly, if AD were joined it could be shown that $\angle$ ABD= $\angle$ ACD,

and $\triangle$ ABD=$\triangle$ACD.

Ex. 62.—**If the opposite sides of a quadrilateral are equal it must be a parallelogram.**

Ex. 63.—**The diagonals of a parallelogram bisect each other.**

Ex. 64.—**If the diagonals of a quadrilateral bisect each other it must be a parallelogram.**

Ex. 65.—**The join of the mid-points of two sides of a triangle is parallel to the base and equal to half of it.**

Ex. 66.—**The straight line through the mid-point of one side of a triangle parallel to another side passes through the mid-point of the remaining side.**

Ex. 67.—**The median line of a triangle divides it into two triangles which are equal in area.**

Ex. 68.—If any point G be taken on the median AD of a $\triangle$ ABC, the $\triangle$s AGB, AGC are equivalent (equal in area). Also if the median AD be produced to any point G, the $\triangle$s AGB, AGC are equivalent.

Ex. 69.—ABC is a $\triangle$, the medians AD, BE intersect in G; show that BGC is one-third of $\triangle$ ABC. Hence, show that median CF also passes through G.

When any number of st. lines pass through the same point they are said to be **concurrent**. Note that '**the three medians of a triangle are concurrent.**'

Ex. 69 (*a*).—**P, Q, R, S are the mid-points of the sides DA, AB, BC, CD of a quadl. ABCD, show that PQRS is a parallelogram, whose sides are parallel to the diagonals of ABCD, and whose area is half that of ABCD.**

Ex. 69 (*b*).—The locus of the mid-point of a st. line OP joining a fixed point O to a point on a fixed st. line AB is a st. line parallel to AB.

Ex. 69 (*c*).—Two ||gms ABCD, AEFD are on the same base AD and on opposite sides of it. If BE, CF be joined, show that BEFC is a ||gm and that it is equal to the sum of ||gms AC, AF.

Ex. 69 (*d*).—In Ex. 69 (*c*) read 'the same side' for 'opposite sides,' and 'difference' for 'sum.'

PROPOSITION 35. THEOREM.

Parallelograms on the same base and between the same parallels are equal to one another.

Let ||gms ABCD, EBCF be on the same base and between the same ||s AF, BC, ABCD shall be equal to EBCF.

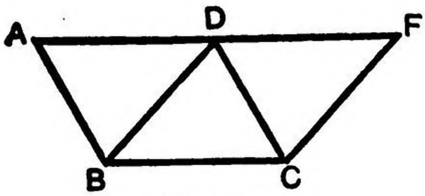


Fig. 1.

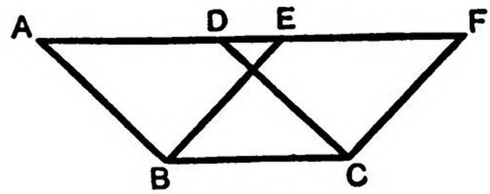


Fig. 2.

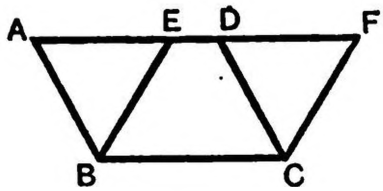


Fig. 3.

If the sides AD, EF be terminated in the same point D (E coinciding with D, Fig. 1), it is plain that each of the ||gms is double of △ BDC, [I. 34.

∴ they are equal. [Ax. 6.

But if the sides AD, EF be not terminated in same point, then ∵ ABCD is a ||gm;

∴ AD=BC. [I. 34.

For the same reason EF=BC,

∴ AD=EF; [Ax. 1.

∴ the whole or remainder AE=whole or remainder DF, [Ax. 2 and 3.

and AB=DC,

and ext. ∠ FDC=int. and opp. ∠ EAB; [I. 29

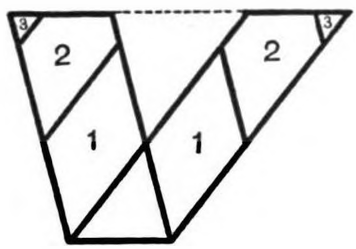
∴ △EAB=△FDC. [I. 4.

If $\triangle$EAB be taken from the figure ABCF the ||gm EBCF remains; if $\triangle$FDC be taken from the figure ABCF the ||gm ABCD remains;

$\therefore$ ||gm ABCD = ||gm EBCF. [Ax. 3.

NOTE.

This is the first instance of two figures being proved by Euclid **equal in area though not necessarily congruent.** Two such figures are sometimes called **equivalent.** The proof, however, depends on congruency.



The accompanying figure indicates how two such ||gms can be divided into *congruent* parts.

Each is divided by lines || to the sides of the other.

Def.—The altitude of a figure, with reference to a given side as base, is the $\perp$r distance between the base and the opposite side or vertex.

Ex. 70.—State and prove a converse of this Proposition.

Ex. 70 (*a*).—If, in one of the figs. I. 35, FK be drawn || to AC meeting BC produced in K. Show that $\triangle$ ABK = quadl. AFCB.

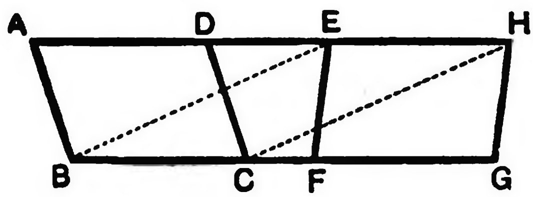
Ex. 70 (*b*).—A trapezoid (*i.e.* a quadl. with two opposite sides parallel) is equal to a triangle between the same parallels on a base equal to the sum of the two parallel sides.

Ex. 70 (*c*).—On the same base BC and on the same side of it construct a ||gm BEFC, equal *in all respects* to a given ||gm ABCD.

PROPOSITION 36. THEOREM.

Parallelograms on equal bases and between the same parallels are equal.

Let ABCD, EFGH be ||gms on equal bases, BC and FG, and between the same ||s, AH and BG; ABCD shall be equal to EFGH.



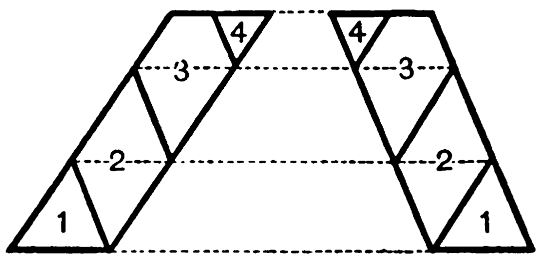
Join BE and CH.

∵ BC = FG,	[HYP.
and FG = EH.	[I. 34.
∴ BC = EH;	[AX. I.
and they are ||.	[HYP.
∴ BE and CH are both equal and ||.	[I. 33.
∴ EBCH is a ||gm.	[DEF.
||gm ABCD = ||gm EBCH on the same base BC, and between the same ||s.	[I. 35.
||gm EFGH = ||gm EBCH on the same base EH, and between the same ||s.	[I. 35.
∴ ||gm ABCD = ||gm EFGH.	[AX. I.

NOTES.

Beginners are apt to assume without proof that the figure EBCH is a parallelogram.

Two such parallelograms on equal bases, and of the same altitude, can always be divided into *congruent* parts in the way indicated in the diagram. Each is divided by lines parallel to the side of the other.



Ex. 71.—State and prove a converse of this proposition.

Ex. 72.—**The locus of a point at a given distance from a given straight line is a pair of parallels on opposite sides of the given straight line.**

Consult the notes on Exx. 7 and 8 as to the meaning of this enunciation.

It is equivalent to two propositions, each the converse of the other.

Ex. 73.—Find the locus of a point equidistant from two given parallel straight lines.

The locus is a certain straight line easily found.

We must demonstrate that—

(1) *Any point on it is equidistant from the parallels.*

(2) *Any point equidistant from the two parallels is on it.*

Ex. 74.—Find points on a given straight line at a given distance from another given straight line.

If there are any points possessing the required property they are at the intersection of the first straight line with a certain locus.

There may be no point at all, but if there is one there must be two.

Ex. 75.—Find points on a given circle at a given distance from a given straight line.

The points required are at the intersection of the circle with a certain locus. (See Ex. 72.)

There may be four, three, or two points on the circle possessing the required property.

There may be only one, or there may be none at all.

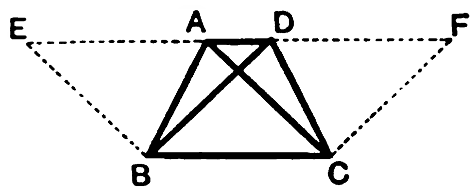
When a problem is like those given as Exx. 74, 75, where there may be several solutions, it is part of the student's duty (whether specially indicated or not) to say what is the greatest number of possible solutions, and to point out the cases in which the full number cannot be obtained.

Ex. 75 (*a*).—ABCD is a given ||gm. On a base FG equal to BC, and in the same st. line with it construct two ||gms EFGH, KFGL, each *equal in all respects* to ABCD.

PROPOSITION 37. Theorem.

Triangles on the same base and between the same parallels are equal to one another.

Let ABC and DBC be △s on the same base BC, and between the same ||s AD and BC, then △ ABC shall be equal to △ DBC.



Produce AD both ways to the points E and F. [Post. 2.
and through B and C draw BE and CF ∥ to CA and BD respectively. [I. 31.

Then EBCA and DBCF are ||gms,
and ||gm EBCA=||gm DBCF. [I. 35.

Now △ ABC is half of the ||gm EBCA, [I 34.

and △ DBC is half of the ||gm DBCF, [I. 34.

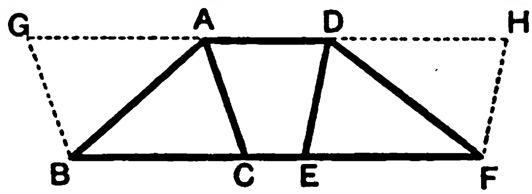
∴ △ ABC=△DBC. [Ax. 7.

Ex. 75 (*b*).—In the fig. of I. 37, show that each of the △s EBD, ACF is equal to the trapezoid ABCD. *Compare* Ex. 70 (*a*) *and* 70 (*b*).

PROPOSITION 38. THEOREM.

Triangles on equal bases and between the same parallels are equal to one another.

Let ABC and DEF be △s on equal bases, BC and EF, and between the same parallels BF and AD, then ABC shall be equal to DEF.



Produce AD both ways to the points G and H, [POST. 2.
and through B and F draw BG and FH ∥ to AC and ED respectively. [I. 31.

Then GBCA and DEFH are ∥gms.
and ∥gm GBCA = ∥gm DEFH. [I. 36.

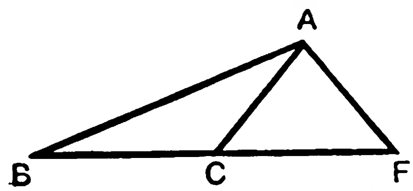
Now △s ABC and DEF are halves respectively of ∥gms GBCA and DEFH. [I. 34.

∴ △ ABC = △ DEF. [AX. 7.

Ex. 75 (*c*).—**The ∥ sides AD, BC of a trapezoid ABCD, are bisected in GH. Show that GH bisects the trapezoid ABCD.**

NOTES.

The particular case of this proposition which is most often wanted is that in which the equal bases are halves of the same line, and the triangles ABC, ACF have one side, AC, common. (See Ex. 76.)

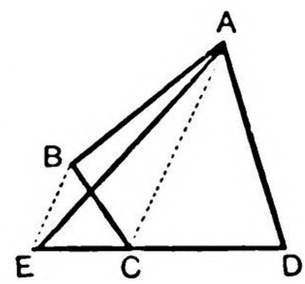


Ex. 76.—If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles contained by those sides supplementary, then the triangles shall be equivalent.

Ex. 77.—Prove the *obverse* of I. 38.

Ex. 78.—To make a triangle equivalent to a given quadrilateral ABCD.

Join any vertex A *with the next but one (either way round) to it,* C. *Through the next vertex* B *draw a parallel to* AC, *meeting the next side to* BC *in* E. AED *shall be the required triangle.*



Ex. 79.—To make a quadrilateral equivalent to a given pentagon.

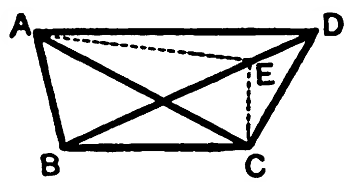
Method similar to that given for the preceding Exercise.

Ex. 80.—To construct a rectilineal figure equivalent to a given rectilineal figure, and having the number of its sides one less than that of the given figure, and thence to construct a triangle equivalent to a given rectilineal figure. (Syllabus.)

PROPOSITION 39. THEOREM.

Equal triangles on the same base and on the same side of it are between the same parallels.

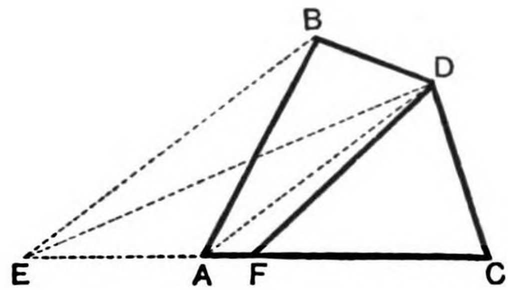
Let ABC and DBC be equal △s on the same base BC, and on the same side of it, they shall be between the same parallels.



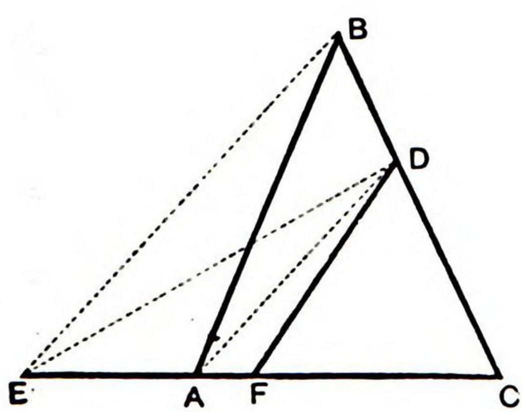
Join AD.
AD shall be ∥ to BC.
For if it is not, through A draw AE ∥ to BC, meeting BD in E, [I. 31.
and join EC.
Then △ ABC = △ EBC. [I. 37.
But △ ABC = △ DBC. [HYP.
∴ △ DBC = △ EBC, [AX. I.
which is impossible.
∴ AE is not ∥ to BC.
In like manner it can be shown that no other straight line through A but AD is ∥ to BC.
∴ AD is ∥ to BC.
(For it has been shown in Prop. 31 that there can always be found one parallel through A to BC.)

NOTES.

Ex. 81.—To bisect a quadrilateral by a line drawn through a given vertex. *If the diagonal* DA *through the given point* D *divides the given quadrilateral into two unequal triangles, through* B, *the vertex of the smaller one, draw a line* BE, *making a triangle,* DEC, *equivalent to the given quadrilateral, and draw its median* DF.



Ex. 82.—To bisect a triangle by a line drawn through a given point in one of its sides. *If the line* DA *joining the given point* D *to the opposite vertex* A *divides the given triangle into two unequal triangles, through* B, *the vertex of the smaller one, draw a line* BE, *making a triangle,* DEC, *equivalent to the given triangle, and draw its median* DF.



(Note the close agreement between this and the previous solution : we treat ABDC as if it were a quadrilateral with D for one vertex.)

From Corollary to I. 37 and I. 39: **'The locus of the vertex of a triangle on a given base, and having a given area, is a pair of straight lines parallel to the given base, and on opposite sides of it.'**

Ex. 83.—ABC and DEF are any two triangles. Find a point P such that the two triangles PBC, PEF are equal respectively to ABC, DEF. P *lies on two different loci, and will be at their intersection if there be such a point.* Show that if there is such a point there must at least be three others possessing the same property, and that there may be an infinite number.

Ex. 84.—If two equivalent triangles ABC, DBC be on the same base BC, and on opposite sides of it, the straight line AD will be bisected by BC or BC produced.

Compare Ex. 68, *of which this is a converse.*

Ex. 84 (*a*).—BA, BD are two given finite st. lines. Show that the locus of a pt. P within the ∠ABD or its vertically opposite angle such that △PAB = PDB is the st. line through B which bisects AD.

What would be the locus if we inserted 'not' before 'within'?

Ex. 84 (*b*).—BA, BD are two given finite st. lines. Show that the locus of a pt. P within the ∠ABD such that the sum of △s PAB, PDB remains constant is a finite st. line parallel to or coincident with BD.

Ex. 84 (*c*).—AB, DC are two given finite st. lines which would meet if produced through A, D in a pt. O. Show that the locus of a pt. P within the ∠BOC such that △PAB = △PDC is a st. line through O.

(*Along* OB, OC *take* OH, OK = AB, DC *respectively, show that* △POH = POK, *and use Ex.* 84 (*a*).)

Ex. 84 (*d*).—AB, DC are two given finite st. lines which would meet if produced through A, D in a pt. O. Show that the locus of a pt. P within the ∠BOC such that the sum of △s PAB, PDC remains constant is a st. line.

(*Along* OB, OC *take* OH, OK = AB, DC *respectively, and use Ex.* 84 (*b*) *to show that* P *lies on a* ∥ *to* HK.)

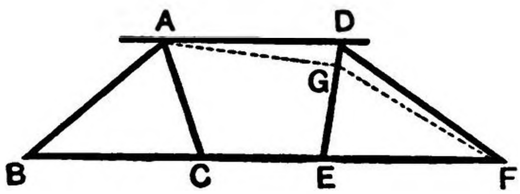
Ex. 84 (*e*).—E and F are the mid-pts. of the diagls. AC, BD of a quadl. ABCD. Show that △AFB + △CFD = half the quadl. ABCD = △AEB + △CED.

Show also by means of 84 (*d*) that if P be any pt. on the st. line EF that △APB + △CPD = half the quadl. ABCD.

PROPOSITION 40. THEOREM.

Equal triangles on equal bases in the same straight line and on the same side of it are between the same parallels.

Let ABC and DEF be equal $\triangle$s on equal bases BC and EF, in the same straight line BF, and on the same side of it, then they shall be between the same parallels.



Join AD. AD shall be $\parallel$ to BF.

For if it is not, through A draw AG $\parallel$ to BF, meeting ED in G, [I. 31.

and join GF.

Then $\triangle$ ABC = $\triangle$ GEF. [I. 38.

But $\triangle$ ABC = $\triangle$ DEF. [HYP.

$\therefore$ $\triangle$ DEF = $\triangle$ GEF. [AX. I.

which is impossible.

$\therefore$ AG is not $\parallel$ to BF.

In like manner it can be shown that no other straight line through A but AD is $\parallel$ to BF.

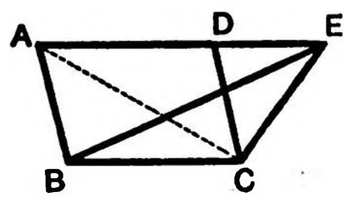
$\therefore$ AD is $\parallel$ to BF.

Ex. 85.—Enunciate, and prove another converse of I. 38.

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let ||gm ABCD and △ EBC be on the same base BC, and between the same parallels BC and AE, then ||gm ABCD shall be double of △ EBC.



Join AC.

Then △ ABC = △EBC. [I. 37.

But ||gm ABCD is double of the △ ABC. [I. 34.

∴ the ||gm ABCD is double of the △ EBC.

Ex. 86.—If two equivalent triangles ABC, DEF be on equal bases BC, EF, in the same straight line BF, and on opposite sides of it, the straight line AD will be bisected by BF.

Ex. 87.—A fixed straight line AD is bisected by any other straight line **BF. On BF are taken any equal segments BC, EF.** Show that the **triangles ABC, DEF are equivalent.**

Ex. 88.—Find a set of lines such that any segment of any one of them is the base of two equivalent triangles, having two given points as vertices.

The set consists of two pencils, one of parallels and the other of concurrent lines.

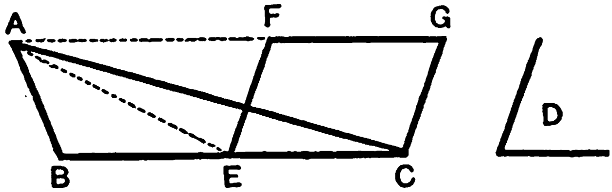
Ex. 88 (*a*).—A point P within ||gm ABCD is joined with A,B,C,D. **Shew that** △ PAB + △ PCD = △ PAD + △ PBC.

Ex. 88 (*b*).—Shew that any number of ||gms can be described each equal to **a given** ||gm, ABCD each having either (1) a side of ABCD for a diagonal or (2) **a diagonal** of ABCD for a side.

PROPOSITION 42. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given △ and D the given rectilineal angle; it is required to describe a ||gm that shall be equal to the given △ABC, and have one of its angles equal to D.



Bisect BC in E [I. 10], and at the point E in the st. line EC make ∠ CEF equal to ∠ D [I. 23]. Through A draw AFG || to EC and through C draw CG || to EF [I. 31]. Then ||gm FECG = △ABC. Join AE. Then ∵ BE = EC. [CONST.

∴ △ABE = △AEC. [I. 38.

∴ △ABC is double of △AEC.

But ||gm FECG is also double of △AEC. [I. 41.

∴ ||gm FECG = △ABC, [AX. 6.

and it has ∠ CEF equal to the given ∠ D. [CONST.

NOTES.

Ex. 89.—To describe a triangle equal to a given parallelogram, and having an angle equal to a given rectilineal angle.

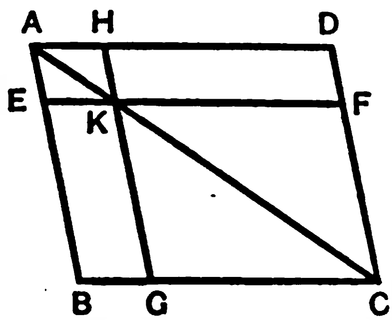
Ex. 90.—Through the extremities of each diagonal of a given quadrilateral a parallel is drawn to the other. Show that the parallelogram thus formed is double the given quadrilateral.

When the angular points of one rectilineal figure are on the sides of another, the first is said to be **inscribed in** the second, and the second is said to be **described about** the first.

PROPOSITION 43. Theorem.

The complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.

Let ABCD be a ||gm, of which the diameter is AC, and EH and GF the ||gms about AC, that is, through which AC passes, and BK and KD the other ||gms which make up the whole figure ABCD, and which are therefore called the complements, then BK shall be equal to KD.



∵ AC is the diagonal of ||gm ABCD.

∴ △ABC = △ADC. [I. 34.

Similarly △AEK = △AHK,

and △KGC = △KFC.

∴ △s AEK and KGC together = △s AHK and KFC. [Ax. 2.

But whole △ABC = whole △ADC.

∴ the remainder, the compt. BK = remainder the compt. KD. [Ax. 3.

Ex. 91.—If HG, EF be two straight lines drawn across a parallelogram ABCD, parallel to the sides AB, AD respectively, and making the parallelograms EG, HF equivalent, show that K lies on the diameter AC.

Where must K be in order that the parallelograms about the diameter may be equivalent as well as the complements?

Ex. 91 (*a*).—A point K is taken on the diagonal AC of a ||gm ABCD and joined with B and D. Shew that $\triangle$ KAB = $\triangle$ KAD and that $\triangle$ KCB = $\triangle$ KCD.

Ex. 91 (*b*).—The diagonal AC of a ||gm ABCD is produced to a point K. If KB, KD be joined, show that $\triangle$ KAB = $\triangle$ KAD and that $\triangle$ KCB = $\triangle$ KCD.

Ex. 91 (*c*).—ABCD is a ||gm. A point K is taken such that $\triangle$ KAB = $\triangle$ KCD, show that K must lie on AC or AC produced.

Enunciate this and the two preceding Exercises as a 'locus' proposition.

Ex. 91 (*d*).—In the fig. of I. 43, a point P is taken on AC or AC produced and joined with B,E,H and D. Show that $\triangle$ PHD = $\triangle$ PEB.

Ex. 91 (*e*).—Enunciate and prove the converse of Ex. 91 (*d*).

Ex. 91 (*f*).—If in the fig. of I. 43 FH, GE be joined and produced to meet in P. Show that $\triangle$ PHD = $\triangle$ PEB and hence that P lies on AC produced.

Ex. 91 (*g*).—Two equiangr. ||gms AEKH, CFKG are placed with equal $\angle$s EKH, FKG vertically opposite. AH and CF are produced to meet in D and KD is joined. If EL, GM be drawn from E, G || to KD and meeting AD, CD in L, M, show that E, L, M, G are the vertices of a |gm equal to the sum of the ||gms EH, GF.

(*This theorem of Pappus was put in the more general form given as* Ex. 156 *by Clavius.*)

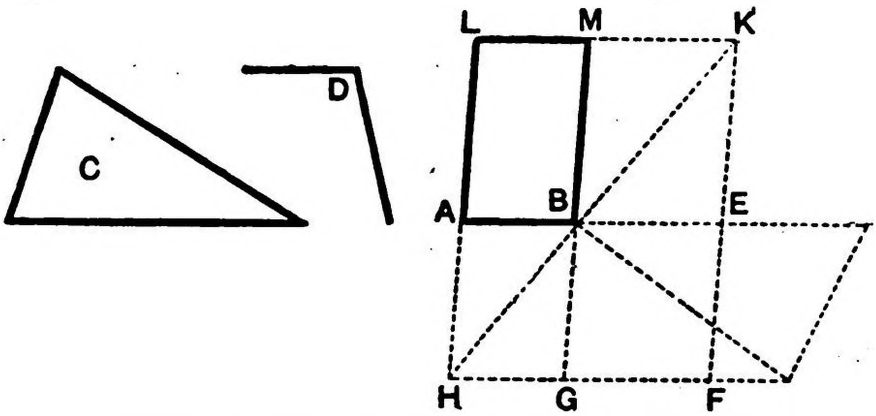
Euclid gave a special name **gnomon** to the figure made up of either of the two ||gms EH, GF about the diameter, AC of the ||gm ABCD together with the two complements BK, HF.

Thus, in the fig. of I. 43, the fig. made up of EH, BK, HF might be called *the gnomon* BHF; and that made up of GF, BK, HF as *the gnomon* BFH.

PROPOSITION 44. PROBLEM.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given st. line, C the given △, and D the given rectl. ∠; it is required to apply to the st. line AB a ||gm equal to △C, and having an ∠ equal to D.



Make ||gm BEFG equal to △C, and having ∠ EBG equal to D, so that BE may be in the same st. line with AB: [I. 22 and I. 42.

produce FG to H, through A draw AH || to BG or EF, [I. 31

and join HB. Then ∵ AH is || to EF,

∴ the two int. ∠s AHF and HFE together = 2 rt. ∠s. [I. 29.

∴ ∠s BHF and HFE are together less than 2 rt. ∠s.

∴ HB and FE will meet if produced towards B and E. [AX. 12.

Let them meet in K, through K draw KL || to EA or FH, [I. 31.

and produce HA and GB to meet KL in L and M.

Then HLKF is a ||gm of which the diameter is HK.

∴ compt. LB=compt. BF. [I. 43.

But BF=△C. [CONST.

∴ BL=△C. [AX. I.

Again ∠ABM=∠GBE. [I. 15.

=D. [CONST.

∴ to the given st. line AB the ||gm LB has been applied equal to △C, and having ∠ABM equal to D.

NOTE.

The student should go carefully through the construction here given by Euclid *practically*, making first of all a triangle congruent with C, and having one of its sides in a straight line with AB; [By I. 22.
next making a parallelogram equivalent to this triangle. [By I. 42.

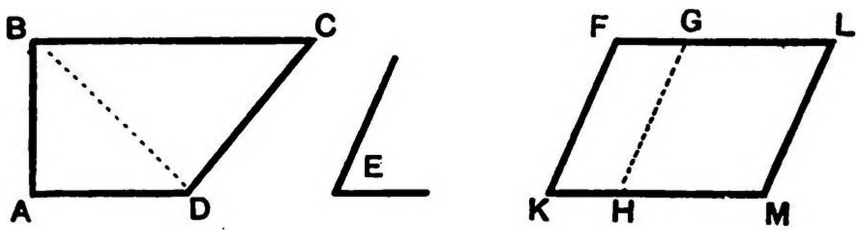
Ex. 91 (*h*).—To make a ||gm equivalent and equiangr. to a given ||gm, and having one side equal to a given st. line.

Ex. 91 (*i*).—Solve I. 44 by making a ||gm equal to △C, etc., but such that BE *falls along* BA.

PROPOSITION 45. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectl. figure, and E the given rectl. ∠; it is required to describe a ||gm equal to ABCD, and having an ∠ equal to E.



Join DB, and describe the ||gm FH equal to △ ADB and having ∠ FKH equal to E. [I. 42.

To the st. line GH apply ||gm GM equal to △DBC, and having ∠ GHM equal to ∠ E. [I. 44.

The figure FKML shall be the ||gm required.

∵ ∠ E = each of ∠s FKH, GHM, [CONST.

∴ ∠ FKH = ∠ GHM. [AX. 1.

∴ ∠s FKH, KHG = GHM, KHG. [AX. 2.

But ∠s FKH, KHG together = 2 rt. ∠s. [I. 29.

∴ ∠s KHG, GHM together = 2 rt. ∠s.

∴ KH is in the same st. line with HM. [I. 14.

Also GL is in the same st. line with FG,

otherwise *two* st. lines FG, GL || to the same st. line KM would meet in G, [I. 30.

which is impossible. [I. 14.

Now ∵ KF || to HG, and HG || to ML, [CONST.

∴ KF || to ML, [I. 30.

and KM and FL are ||,

∴ KFLM is a ||gm. [DEF. 36.

And ∵ △ **ABD** = ||gm **HF** (CONST.), and △ **BDC** = ||gm **GM** (CONST.),

∴ the whole rectl. figure **ABCD** = whole ||gm **KFLM**. [AX. 2.

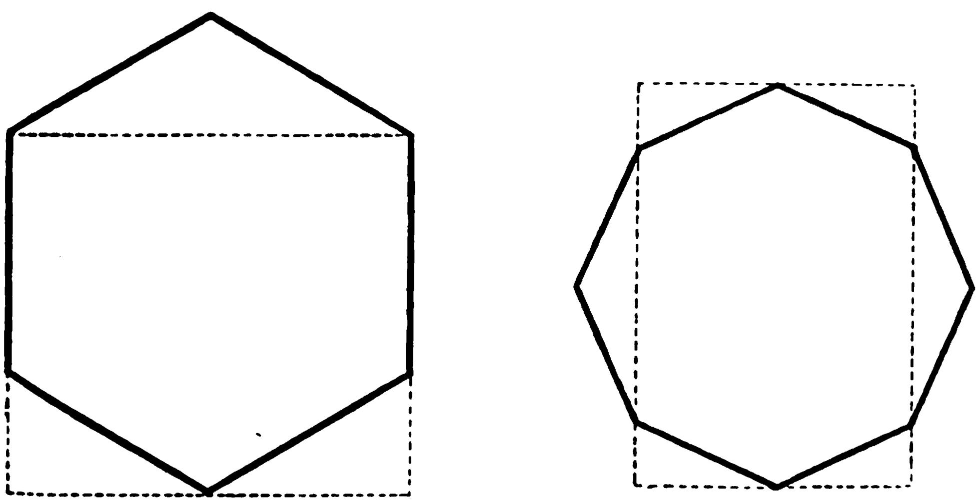
∴ the ||gm **KFLM** has been described equal to the given rectl. figure **ABCD**, and having **FKM** equal to **E**.

COROLLARY.—**From this it is manifest how to a given st. line to apply a parallelogram which shall have an angle equal to a given rectilineal angle, and shall equal a given rectilineal figure** (*by applying to the given st. line a parallelogram equal to the first* △ ABD *and having an angle equal to the given angle, and so on*). [I. 44.

NOTE.

It is frequently unnecessary to divide a given rectilineal figure into triangles in order to obtain an equivalent parallelogram.

A construction is indicated by the annexed diagram for obtaining a parallelogram equivalent to a given regular hexagon or octagon.



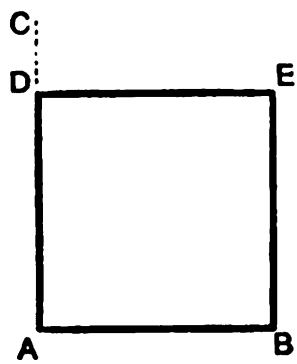
In practice with an irregular figure, it would be found less tedious to use the method of Ex. 78, 79 to get a *triangle* equivalent to the given rectilineal figure, and then use I. 42.

Def.—A square is a four-sided figure which has all its sides equal and all its angles right angles (*i.e.* which is *equilateral and rectangular*).

PROPOSITION 46. Problem.

To describe a square on a given straight line.

Let AB be the given st. line; it is required to describe a sq. on AB.



From pt. A draw AC perp. to AB, [I. 11.
and make AD equal to AB. [I. 3.
Through D draw DE ∥ to AB, and through B draw BE ∥ to AD. [I. 31.

ADEB shall be a square.

Now ADEB is a ∥gm. [const.
∴ AB = DE, and AD = BE. [I. 34.
But AB = AD. [const.
∴ the four st. lines BA, AD, DE, EB = one another. [Ax. 1.

Again ∵ DE ∥ to AB,
∴ the two int. ∠s BAD, ADE together = 2 rt. ∠s. [I. 29.
But BAD is a rt. ∠. [const.
∴ ADE is a rt. ∠. [Ax. 3.

∴ each of the opposite ∠s ABE and BED is a rt. ∠. [I. 34.

∴ ADEB is rectangular,

and it has been proved to be equilateral.

∴ it is a square, and it has been described on the given st. line AB.

COROLLARY.—**Hence every parallelogram which has one angle a right angle has all its angles right angles.**

Such a parallelogram is called a **rectangle.**

Ex. 92.—To describe a rhombus on a given straight line, and having an angle equal to a given rectilineal angle.

Ex. 92 (*a*).—Describe a rectangle with its adjacent sides equal to two given st. lines.

Ex. 92 (*b*).—ABML is a rectangle. Through B any st. line HBK is drawn cutting LA, LM produced in H, K (See fig. of I. 44). Show that ABML=rect. whose adjacent sides are equal to AH, MK.

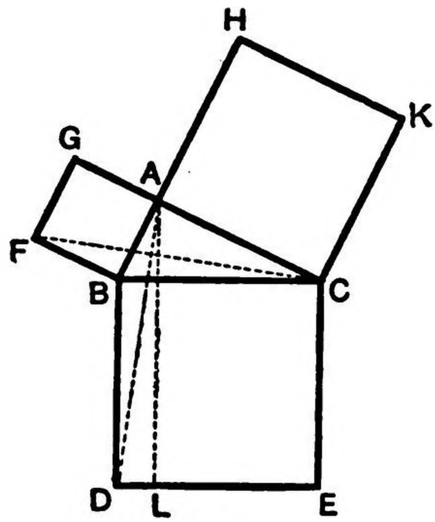
Hence give a construction for describing on a given st. line a rectangle equal to a given rectangle.

DEF.—**A triangle which has a right angle is called 'a right-angled triangle.'**

PROPOSITION 47. THEOREM.

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides containing the right angle.

Let ABC be a rt.-angled △, having the rt. ∠ BAC: the sq. described on the side BC shall be equal to the sqs. described on BA, AC.



On BC, CA, AB describe respectively the sqs. BDEC, CKHA, AGFB. [I. 46.

Through A draw AL ∥ to DB or CE. [I. 31.

Join AD, FC.

∵ BAC is a rt. ∠, [HYP.

and BAG is a rt. ∠, [CONST.

∴ CA and AG are in the same st. line. [I. 14.

Now the rt. ∠ DBC = rt. ∠ FBA, [AX. 11.

∴ whole ∠ DBA=whole ∠ FBC. [AX. 2.

Hence in △s ABD, FBC, AB, BD = FB, BC, each to each, [DEF. 30.

and ∠DBA = ∠FBC,

∴ △ABD = △FBC. [I. 4.

Now ||gm BL is double of △ABD, [I 41.

and sq. GB is double of △FBC, [I 41.

∴ ||gm BL = sq. GB. [AX. 6.

In like manner, by joining AE, BK, it can be proved that ||gm CL = sq. CH.

∴ whole sq. BDEC = the two sqs. GB and HC. [AX. 2.

i.e. sq. on BC = sqs. on BA, AC.

Ex. 92 (*c*).—In the fig. of I. 47, instead of joining AD, FC, produce DB, LA to meet FG produced in M, N and show that ||gm BL = ||gm BN = sq. GB. Hence obtain another proof of I. 47.

Ex. 92 (*d*).—In the fig. of I. 47, instead of describing the sq. BDEC on the opposite side of BC to A, describe it on the same side, and show that D lies on FG or FG produced. Hence show that ||gm BL = twice △BAD = sq. GB, and obtain another proof of I. 47.

In connection with I. 47, the following diagrams may be found useful by those teachers who try to make their pupils realise the truths of the theorems demonstrated in the 'Elements' by *Experimental Geometry*. Figs. 1 and 2 show how a square may be made equal to five times a given square; fig. 3 how to make one equal to ten times a given square.

FIG 1.

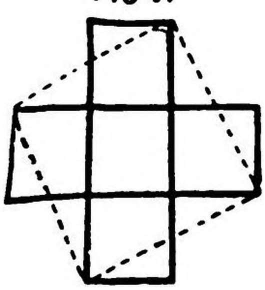


FIG 2.

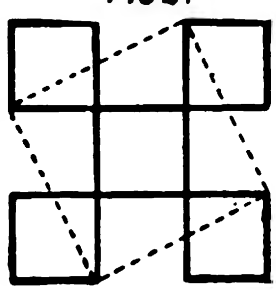
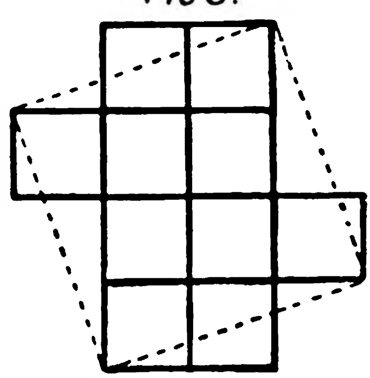


FIG 3.



NOTES.

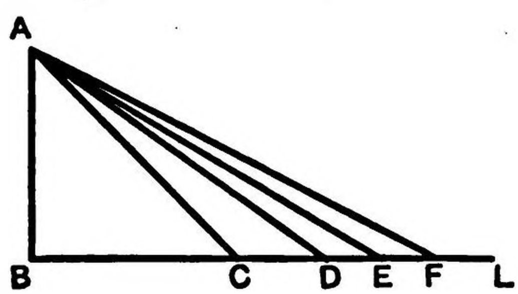
We have not only proved the Theorem enunciated, but incidentally the following highly important one :—

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the square on either of the sides containing the right angle is equal to the rectangle contained by the base and its segment adjacent to that side.

(*Rectangle* BL = *sq. on* AB.)

Ex. 93.—Construct a square whose area shall be twice, three times, four times . . . a given square.

Make ABL a right angle.
Make BC=AB,
BD=AC,
BE=AD,
BF=AE, and so on;



then the areas of the squares on AC, AD, AE, AF . . . shall be twice, three times, four times . . . the area of the squares on AB.

If the area of a square be two square inches we may speak of its side as being $\sqrt{2}$ of an inch. Hence the above method shows how to represent $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc., geometrically.

The student has perhaps noticed that

$$3^2+4^2=5^2,$$
$$5^2+12^2=13^2,$$

and *assuming that the number of units of area in a square is the square of the number of linear units in its side*, has deduced that if the sides containing the right angle are 3 and 4 the hypotenuse will be 5, while if they are 5 and 12 the hypotenuse will be 13, and may have wondered whether any general method existed for determining similar sets of numbers.

The following tables supply two different methods of calculating an unlimited number of such :—

TABLE I.

Write down the natural numbers,	1,	2,	3,	4,	5,	6,	7,	8
Multiply each by 4, . . .	4,	8,	12,	16,	20,	24,	28,	32
Add one to the product of each of these and the one above, .	5,	17,	37,	65,	101,	145,	197,	257
Subtract 2 from each, . .	3,	15,	35,	63,	99,	143,	195,	255

Each column below the line gives three numbers with the required property.

The student if acquainted with Algebra may see that this table gives the result of substituting

1, 2, 3 . . . for n in the identity $(4n^2-1)^2+(4n)^2=(4n^2+1)^2$.

TABLE II.

Write down the natural numbers, . .	1,	2,	3,	4,	5,	6,	7
Under each write the sum of the series, .	1,	3,	6,	10,	15,	21,	28
Multiply each by 4, ...	4,	12,	24,	40,	60,	84,	112
Add 1 to each, ...	5,	13,	25,	41,	61,	85,	113
Write down the odd numbers beginning with 3, ...	3,	5,	7,	9,	11,	13,	15

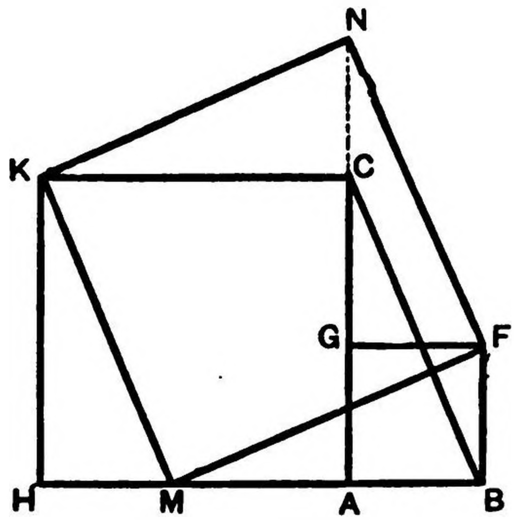
Each column below the line gives three numbers with the required property.

They might be obtained also by substituting

1, 2, 3 . . . for n in the identity $\{2n(n+1)\}^2+(2n+1)^2=(2n^2+2n+1)^2$.

Many proofs of this theorem by dissection and superposition have been given, of which, perhaps, the following are best deserving of notice :—

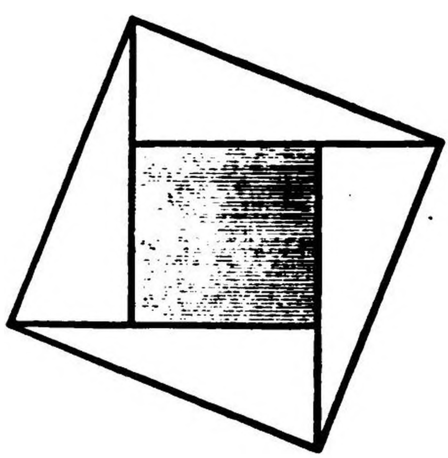
Ex. 94.—Describe the squares ABFG and AHKC so as to have AB and AH in the same straight line. Take M in HB and N in AC produced, such that HM and CN each = AB. Then the four triangles KCN, FGN, FMB, HMK are congruent. Take away the first two from the whole

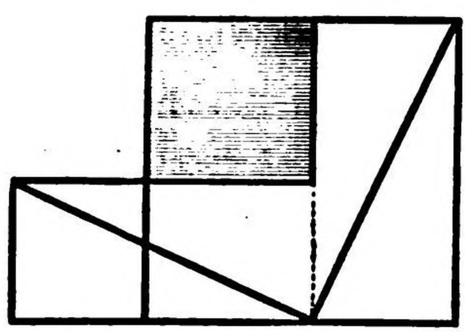


figure, and we get the two squares. Take away the other two, and we get the square on a line equal to the hypotenuse (the side opposite the right angle).

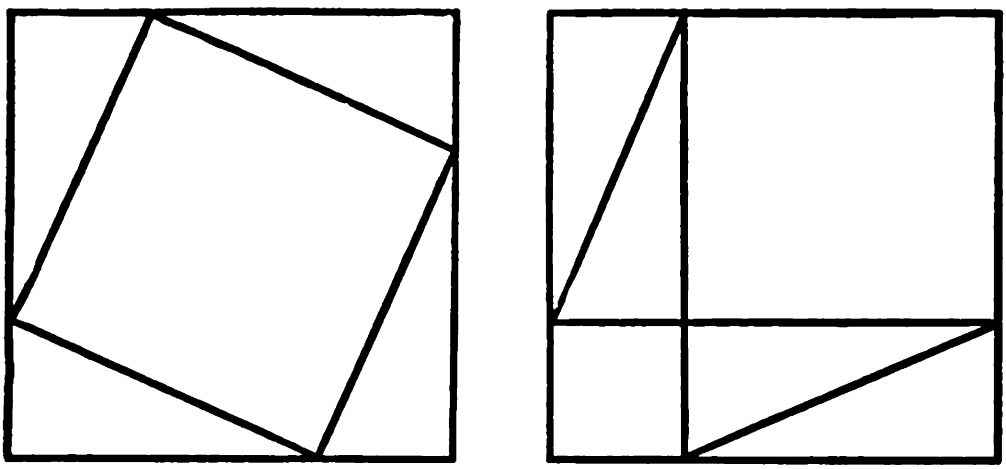
The proof is left to the student.

Ex. 95.—Four congruent right-angled triangles may be arranged around the square on the difference of the two sides, so as to make with it the square of the hypotenuse; they may also be arranged so as to make with it the squares on the two sides.





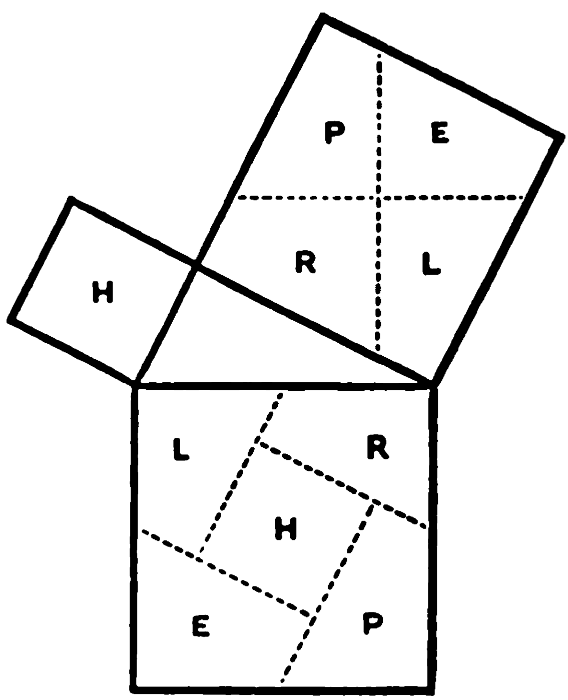
Ex. 96.—Four congruent right-angled triangles may be removed from the square on the sum of the two sides containing the right angle, so as to leave the square on the hypotenuse; they can also be removed from the same square so as to leave the squares on the two sides.



The student should cut out pieces of cardboard of the shapes, and arrange them in the several ways, here indicated.

H. PERIGAL'S DISSECTION.

This is, we believe, by far the simplest and most elegant of all. The student is left to discover the constructions and to prove the geometrical theorems involved in it.

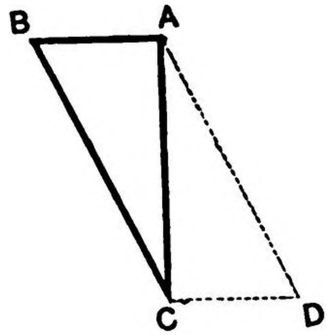


Ex. 96 (*a*).—Show that the **sets of numbers** given by the **formulæ on p. 91** are included **in the** more general formula $(a^2-e^2)^2+(2ae)^2=(a^2+e^2)^2$.

PROPOSITION 48. Theorem.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides, the angle contained by these two sides is a right angle.

Let the sq. described on BC, one of the sides of △ ABC, be equal to the sqs. described on the other two sides BA, AC, then BAC shall be a rt. angle.



From C draw CD perp. to AC, [I. 11.
making CD equal to AB. [I. 3.
Join DA.

Sq. on AD = sqs. on CA, CD, [I. 47.
= sqs. on CA, AB, [∵ CD = AB.
= sq. on CB; [Hyp.
∴ AD = CB.

Again BA = DC, [Const.
and AC is common to the two △s BAC and DAC,
and BC = DA. [Demon.

∴ ∠ BAC = DCA, [I. 8.
which is a rt. angle. [Const.

NOTES.

We have here a converse proved directly.

Ex. 97.—Squares on equal straight lines are equal. (Superposition.)

Ex. 98.—The square on the greater of two unequal straight lines is greater than the square on the less.

Ex. 99.—Equal squares are on equal straight lines. (Immediate reference from 98.)

Ex. 100.—Two congruent rectangles can be placed, with the square on the difference of their sides, so as to make up the squares on the two sides.

Ex. 101.—Two congruent rectangles, together with the squares on their **sides, can be placed so as** to make the square on the sum of the two sides.

Ex. 102.—If we read 'greater than' for 'equal to' in the Enunciation **of I. 48, we must read** 'obtuse' for 'right.'

Ex. 103.—Make corresponding substitution for 'less,' 'acute.'

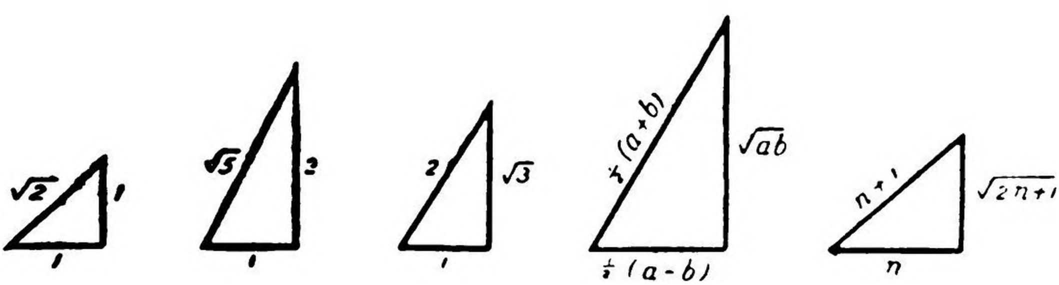
Ex. 104.—State and prove the converses of 102, 103.

Ex. 104 (*a*).—Show that the method of Ex. 93 may be used for solving the problem 'To make a square equal to any number of given squares.'

Ex. 104 (*b*). Show how to make a square equal to the difference of two given squares.

The following Arithmetical and Algebraical results are worthy of **notice :—**

If the sides of a triangle are $(1, 1, \sqrt{2})$, $(2, 1, \sqrt{5})$, $(2, 1, \sqrt{3})$, $\left(\frac{a+b}{2}, \frac{a-b}{2}, \sqrt{ab}\right)$ or $(n, n+1, \sqrt{2n+1})$, the triangle is right-angled.



DEFINITIONS, POSTULATES, AND AXIOMS.

For the sake of reference, and for examination purposes, a complete set of Definitions, Postulates, and Axioms is appended.

DEFINITIONS.

1. *A Point is that which has no parts, or which has no magnitude.*

2. *A Line is length without breadth.*

3. *The Extremities of a Line are points.*

4. *A Straight Line is that which lies evenly between its extreme points.*

5. *A Superficies is that which has only length and breadth.*

6. *The Extremities of a Superficies are lines.*

7. *A Plane Superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.*

8. *A Plane Angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.*

9. *A Plane Rectilineal Angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.*

10. *When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a Right Angle; and the straight line which stands on the other is called a Perpendicular to it.*

11. *An Obtuse Angle is that which is greater than a right angle.*

12. *An Acute Angle is that which is less than a right angle.*

13. *A Term or Boundary is the extremity of anything.*

14. *A Figure is that which is inclosed by one or more boundaries.*

15. *A Circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.*

16. *And this point is called the Centre of the circle.*

17. *A Diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.*

18. *A Semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.*

19. *A Segment of a circle is the figure contained by a straight line and the circumference it cuts off.*

20. *Rectilineal Figures are those which are contained by straight lines.*

21. *Trilateral Figures, or Triangles, by three straight lines.*

22. *Quadrilateral, by four straight lines.*

23. *Multilateral Figures, or Polygons, by more than four straight lines.*

24. *Of three-sided figures, an Equilateral Triangle is that which has three equal sides.*

25. *An Isosceles Triangle is that which has only two sides equal.*

26. *A Scalene Triangle is that which has three unequal sides.*

27. *A Right-Angled Triangle is that which has a right angle.*

28. *An Obtuse-Angled Triangle is that which has an obtuse angle.*

29. *An Acute-Angled Triangle is that which has three acute angles.*

30. *Of four-sided figures, a Square is that which has all its sides equal, and all its angles right angles.*

31. *An Oblong is that which has all its angles right angles, but has not all its sides equal.*

32. *A Rhombus is that which has all its sides equal, but its angles are not right angles.*

33. *A Rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.*

34. *All other four-sided figures besides these are called Trapeziums.*

35. *Parallel Straight Lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.*

36. *A Parallelogram is a four-sided figure whose opposite sides are parallel, and the Diameter (or Diagonal) is the straight line joining two of its opposite angles.*

POSTULATES.[1]

Let it be granted :—

1. *That a Straight Line may be drawn from any one point to any other point.*

2. *That a Terminated Straight Line may be produced to any length in a straight line.*

3. *And that a Circle may be described from any centre, at any distance from that centre.*

[1] *i.e.* Elementary Problems whose construction it is to be taken for granted we can effect.

AXIOMS.[1]

1. *Things which are equal to the Same are Equal to One Another.*

2. *If equals be Added to equals, the Wholes are equal.*

3. *If equals be Taken from equals, the Remainders are equal.*

4. *If equals be Added to unequals, the Wholes are unequal.*

5. *If equals be Taken from unequals, the Remainders are unequal.*

6. *Things which are Double of the same are Equal to one another.*

7. *Things which are Halves of the same are Equal to one another.*

8. *Magnitudes which Coincide with one another, that is, which exactly fill the same space, are Equal to one another.*

9. *The Whole is greater than its Part.*

10. *Two Straight Lines cannot inclose a Space.*

11. *All Right Angles are equal to one another.*

12. *'If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.'*

[1] *i.e.* Elementary Theorems whose truth is taken for granted.

This distinction of postulate and axiom as *problem* and *theorem* which it is now usual to make, was not made by Euclid, with whom all *geometrical demands* (including axioms 10, 11, 12) were postulates. His axioms were thus merely *common notions*—'notions common to all kinds of magnitude as well as space magnitude.'—De Morgan.

PROPERTIES OF TRIANGLES.

Ex. 105.—**The three straight lines which bisect at right angles the sides of a triangle meet at a point.**

The ⊥r *bisectors of the sides* BC, CA *of a* △ ABC *meet at a point* O, *which can be shown to be equidistant from* A *and* B, *and is* ∴ *on the* ⊥r *bisector of* AB.

Note.—*This point is equidistant from* A, B, *and* C (*so that a circle can be described to pass through* A, B, *and* C), *and is called the* circum-centre *of the triangle.*

Ex. 106.—**The three straight lines which bisect the interior angles of a triangle meet in a point.**

The point in which any two of them meet can be shown to lie on the third. (See Ex. 36.)

Note.—*This point is equidistant from the three sides, and is called the* **in-centre** *of the triangle.*

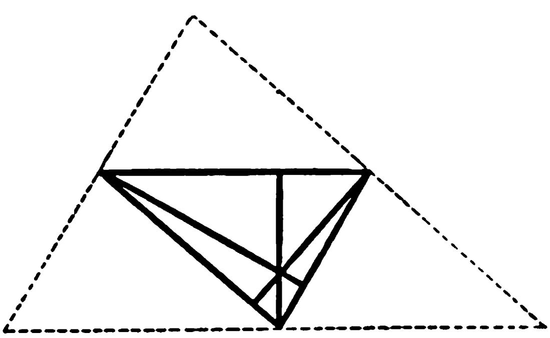
Ex. 107.—**The three median lines of a triangle meet in a point.**

Prove by two applications of Ex. 65.

Note.—*This point is a point of trisection for each median, and is called the* **centroid** *of the triangle.*

Ex. 108.—**The three perpendiculars from the angular points of a triangle to the opposite side meet in a point.**

Through each angular point draw a parallel to the opposite side. The three perpendiculars can be shown to bisect at right angles the sides of the new triangle thus described about the first, and ∴, *as in* Ex. 105, *to meet in a point. This point is called the* ortho-centre *of the triangle.*



Ex. 109.—**Any straight line drawn across a triangle parallel to one side is bisected by the median which bisects that side.**

Prove indirectly that the point at which the straight line cuts the median is its mid-point.

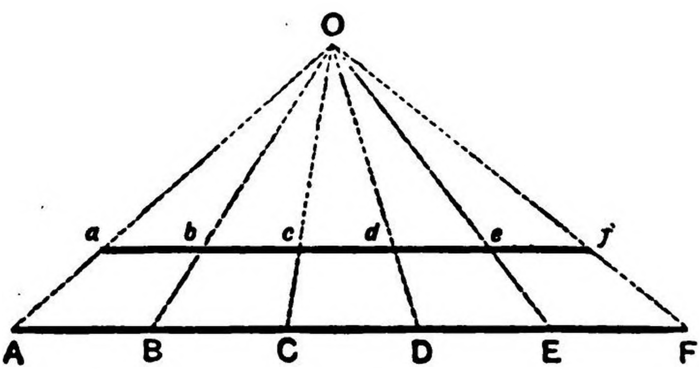
MISCELLANEOUS EXERCISES—I.

Ex. 110.—ABC, DEF are two triangles on equal bases, BC, EF, and between the same parallels, AD, BF. Any straight line parallel to BF cuts AB, AC, DE, DF, in G, H, K, L. Show that GH = KL.

Prove directly that △ AGC = △ DEL, *and hence indirectly that* GH = KL.

NOTE.—An important special case of this is given as Ex. 109.

Ex. 111.—AF is a straight line divided into equal parts, AB, BC, CD, DE, EF. O is any external point.



If OA, OB, OC, OD, OE, OF (or these lines produced) meet a straight line parallel to AF in a, b, c, d, e, f, show that ab, bc, cd, de, ef, are equal to one another. Hence obtain a practical method of dividing a given straight line into any number of equal parts.

Prove any two of them equal by the method indicated for Ex. 110.

Ex. 112.—In the figure of I. 43 prove that EH is parallel to FG.

Prove by I. 39 *that each is parallel to* BD.

Ex. 113.—If in the same figure GE, CA, FH be produced they all pass through one point.

Use Ex. 63 *and* 109, *and prove indirectly.*

ON QUADRILATERALS

(See DEFINITIONS 30-34, and HENRICI, *Congruent Figures*, pp. 113-124).

The word *oblong* is not often used. Those properties of the oblong to which we want to draw attention in investigating the nature of some figure in which it occurs are usually those which belong to it solely in virtue of its being rectangular, and therefore those which it shares with the square. It is convenient therefore to have a name which will apply at once to such a figure and to a square. The word *rectangle* has been chosen to denote a rectangular quadrilateral. (*See* Cor. I. 46.)

In modern works on Geometry it is usual to employ the word *rhombus* in a wider sense than that assigned to it by Euclid. It is used to denote any quadrilateral with four equal sides. Those properties of Euclid's rhombus to which we want to draw attention in investigating the nature of some figure in which it occurs are usually those which belong to it solely in virtue of its four sides being equal, and therefore those which it shares with the square. It is convenient therefore to extend the signification of the name so as to make it include all figures which have the properties it is chiefly used to draw attention to.

The word *rhomboid* is not often used. Those properties of a rhomboid to which we want to draw attention in investigating the properties of some figure where it occurs are just those which it shares with the square, the oblong, and the rhombus. These figures and the rhomboid are all included under the name parallelograms.

Euclid has classified those quadrilaterals which he has deemed worthy of special names thus :—

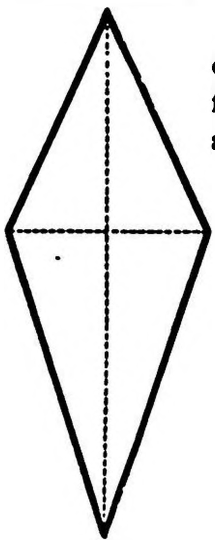
A. Right-angled.		B. Not right-angled.	
(1) Equilateral. (Square.)	(2) Not Equilateral. (Oblong.)	(1) Equilateral. (Rhombus.)	(2) Not Equilateral. (Rhomboid.)

Any other quadrilateral he treats as unworthy of separate study, and says it is called a *trapezium*.

Each of the classified figures is seen to be a parallelogram (Class A by I. 28; Class B by I. 8 and I. 28), and conversely, a parallelogram must belong to one of the four classes of figures.

We may say that he divides all quadrilaterals into *parallelograms* and *trapezia*, the former being further classified, the latter not.

It will, however, be found that some of these trapezia are important figures, worthy of special study, and therefore of distinguishing names.



Consider for instance the figure formed by the sides of two isosceles triangles on a common base. This figure occurs in the diagrams to Propositions 5 and 9, and can be seen to have the following properties :—

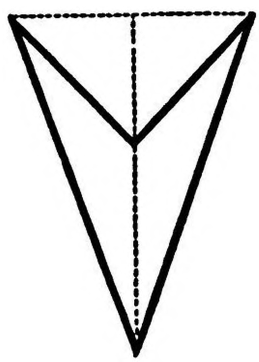
1st. One diagonal is the perpendicular bisector of the other.

2d. The axis bisects the angles at the vertices which it joins.

3d. The other two angles are equal.

4th. The axis divides it into two congruent triangles, with equal sides adjacent.

5th. The median lines meet on the axis, and are equally inclined to it.

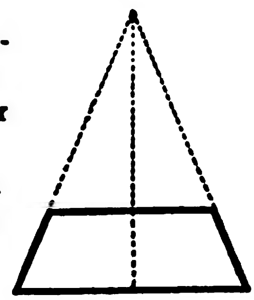


Again, consider the quadrilateral formed by drawing from any point in the side of an isosceles triangle a straight line parallel to the base. Such a figure can be easily shown to be symmetrical about the line bisecting the

vertical angle of the isosceles triangle, and further examination will show it to possess the following properties :—

1st. Two opposite sides have a common perpendicular bisector.

2d. The other two opposite sides are equal, and equally inclined to either of the other sides.



3d. Each angle is equal to one, and supplementary to the other, of its two adjacent angles.

4th. The diagonals are equal, and divide each other equally.

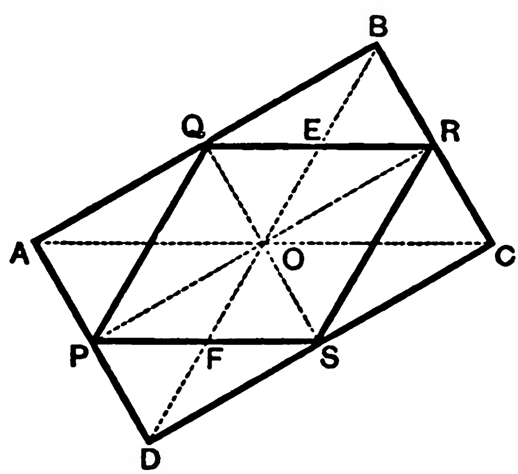
5th. The one median line bisects the angle between the two diagonals, and likewise the angle between those two sides produced which it does not bisect.

6th. The other median line bisects the two diagonals, and is parallel to the two sides which it does not bisect.

7th. The two median lines are each the perpendicular bisector of the other.

We are led then to attempt a more complete classification of quadrilaterals, and we shall base it on the connection of the parallelogram which can be formed by joining the mid-point of each side of any quadrilateral to that of the next with various other lines in the figure.

Let ABCD be a quadrilateral; P, Q, R, S the mid-points of its sides; let the lines PR, QS (called the medians of the quadrl.) intersect in O, then PQRS is always a paralellogram, with each side parallel either to AC or BD. (*See* Ex. 65).



Let BD cut QR and PS in E and F.

I. (*a.*)—Let each of the diagonals AC, BD pass through O.

Since O is the intersection of the diagonals PR and QS of ||gm PQRS,

∴ EO = OF. [Ex. 63.

∵ P and Q are the mid-points of DA, AB, and PF and QE are parallel to AC,

$\left.\begin{array}{r}\therefore \text{BE}=\text{EO},\\ \text{and OF}=\text{FD}.\end{array}\right\}$ [Ex. 66.

∴ BO = OD.

Similarly AO = OC.

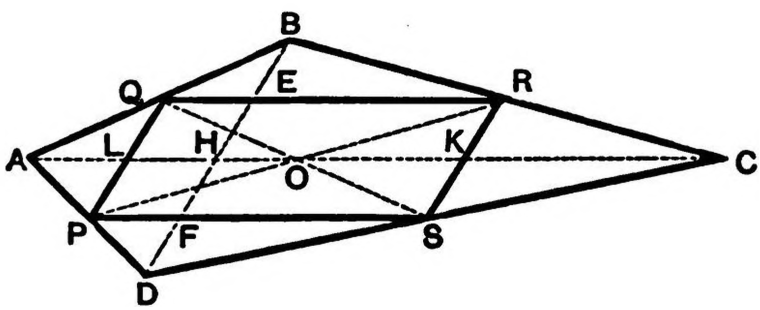
∴ ABCD is a ||gm. [Ex. 64.

If PQRS is a *rhombus* ABCD will be a *rectangle.*
If PQRS is a *rectangle* ABCD will be a *rhombus.*

(*b.*)—Conversely, if a quadrilateral ABCD be a parallelogram, each of its diagonals will pass through O, the intersection of its medians PR, QS.

If ABCD be a *rectangle* PQRS will be a *rhombus.*
If ABCD be a *rhombus* PQRS will be a *rectangle.*

II. (*a.*)—Let one diagl. AC pass through O and cut the other BD at some other point H.



Then (I. *b*) the figure is not a ||gm. Let AC cut PQ and RS in L and K.

BH = twice EH,
= twice QL,
HD = twice HF,
= twice PL,
∴ AC bisects BD;

and ∴ AC bisects all st. lines drawn across the figure parallel to BD. [Ex. 109.

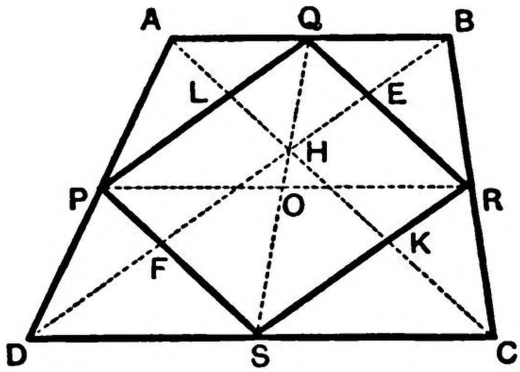
If PQRS be a rectangle AB = AD,
and CB = CD,
and ABCD will have the properties enumerated on p. 102.

(*b.*)—Conversely, if one diagonal AC of a trapezium bisect the other BD at H, it passes through the intersection O of its medians PR, QS.

For as before BH = twice EH,
and HD = twice HF,
∴ EH = HF;
∴ AC passes through O.

If AB = AD,
and CB = CD,
PQRS will be a rectangle.

III. (*a.*)—Let neither of the diagls. AC, BD pass through O, but let their point of intersection at H be on the median QS.



Then ABCD cannot belong either to Class I. (by I. *b*) or to Class II. (by II. *b*).

But as before E and L are mid-points of BH, AH,
and ∴ EL ∥ AB,
and similarly FK ∥ CD.

But since H is on the diagl. QS of the ∥gm PQRS,
EL ∥ FK, [Ex. 112.
∴ AB ∥ CD.

∴ all lines drawn across the figure parallel to PR are bisected by QS. [Ex. 109.

If PQRS is a *rhombus*, ABCD will have axial symmetry about QS, and ∴ will have all the properties enumerated on page 103.

(*b.*)—Conversely, if a trapezium ABCD have AB parallel to CD, then the intersection H of the diagonals AC, BD will lie on the median QS.

For as before EL ∥ AB,
and FK ∥ CD,
∴ EL ∥ FK;
∴ H lies on QS; [Ex. 91.
and if ABCD has axial symmetry about QS,
PQRS will be a rhombus.

IV.—Let neither of the diagls. AC, BD pass through O, and let H not lie on either of the medians.

Then ABCD cannot belong to Class I. (by I. *b*), to Class II. (by II. *b*), or to Class III. (by III. *b*).

It cannot have all lines drawn parallel to a diagonal or a median bisected.

It may still, however, have some important properties connected with the circle, the discussion of which requires a knowledge of the properties of that figure.

Our classification of quadrilaterals in general is therefore as exhaustive as Euclid's of parallelograms. But as we have seen the inconvenience of his set of names for his mutually exclusive classes of figures (see p. 101), it will be well to avoid a similar one in giving names to our own.

The properties of the particular trapezium discussed on p. 102 are those which belong to it solely in virtue of its being symmetrical about a diagonal. These it shares with equilateral quadrilaterals. Let all such figures as are symmetrical about a diagonal be called **kites**.

Again the properties of the particular trapezium discussed on p. 103 are those which belong to it solely on account of its being symmetrical about a median. These it shares with the rectangular quadrilaterals. A name is wanted to include all such figures. Prof. Henrici uses **symmetrical trapezium**.

We doubt whether common usage justifies us in extending the signification of **trapezium** to any sort of *parallelograms*, and though, as is the case of the **rhombus**, the extension might after a time be allowed, the length of its designation is against its general adoption.

Following the analogy of the word **kite**, we venture to suggest **axe** or **axe-head** for all such figures as are symmetrical about a median.

Def.—A quadrilateral ABCD having one pair of parallel sides is often called a trapezoid.

ON LOCI. (SYLLABUS.)

DEF.—If any and every point on a line, part of a line, or group of lines (straight or curved), satisfies an assigned condition, and no other point does so, then that line, part of a line, or group of lines is called the locus of the point satisfying that condition.

In order that a line or group of lines X *may be properly termed the locus of a point satisfying an assigned condition* A, *it is necessary and sufficient to demonstrate the two following associated Theorems:—*

(1.) *If a point satisfies* A, *it is upon* X; or, *if a point is not upon* X, *it does not satisfy* A.

(2.) *If a point is upon* X, *it satisfies* A; or, *if a point does not satisfy* A, *it is not upon* X.

1. The locus of a point at a given distance from a given point is the circumference of a circle having a radius equal to the given distance and its centre at the given point.
2. The locus of a point at a given distance from a given straight line is the pair of straight lines parallel to the given line, at the given distance from it and on opposite sides of it.
3. The locus of a point equidistant from two given points is the straight line that bisects, at right angles, the line joining the given points.
4. The locus of a point equidistant from two intersecting straight lines is the pair of lines, at right angles to one another, which bisect the angles made by the given lines.

INTERSECTION OF LOCI.

If X is the locus of a point satisfying the condition A, and Y the locus of a point satisfying the condition B; then the intersections of X and Y, and these points only, satisfy both the conditions A and B.

1. There is one and only one point in a plane which is equidistant from three given points not in the same straight line.
2. There are four and only four points in a plane each of which is equidistant from three given straight lines that intersect one another but not in the same point.

ON SOLVING GEOMETRICAL PROBLEMS.

I.—METHOD OF INTERSECTION OF LOCI.

Many problems propose, directly or indirectly, *the determination of a point which satisfies two given conditions.*

Such a problem is best attacked by the Method of Intersection of Loci. The student should proceed as follows:—

Consider one *of the given conditions only, and find, if possible, the locus of a point satisfying it.*

Next consider the other given condition by itself, and find, if possible, the locus of a point satisfying it.

Any point in which the two loci may happen to intersect satisfies each of the given conditions.

1. *If the two loci do not intersect there is no such point, and the solution of the proposed problem is impossible.*

2. *If the two loci intersect in more than one point there is more than one solution of the proposed problem.*

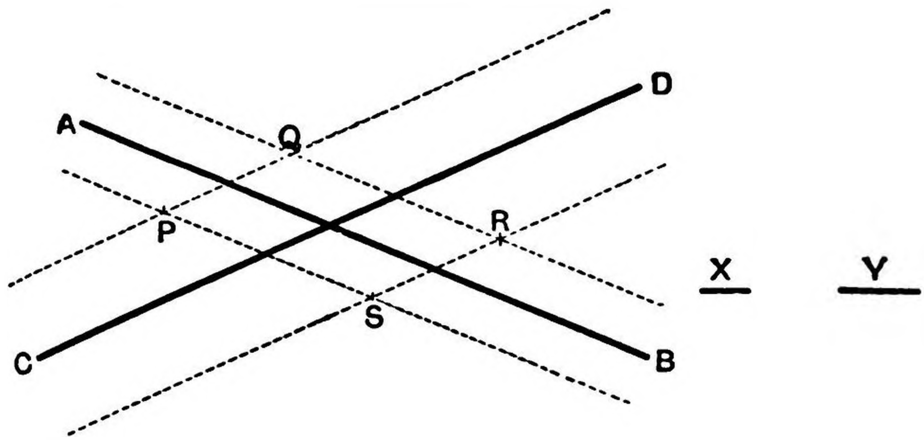
3. *If a line forming the whole or part of one locus coincides with a line forming the whole or part of the other, every point on such a line satisfies both conditions, and the problem is an indeterminate one.*

The *complete* solution of a problem includes an investigation—(1) *Of the number of possible solutions*, and (2) *Of the particular conditions under which the problem may become impossible or indeterminate.*

As an instance of the application of the method, take the following problem:—

To find a point which shall be at given distances from two given straight lines.

Let AB, CD be the two given straight lines, and let the point be re-



quired to be at a distance from AB equal to the given straight line X, and at a distance from CD equal to the given straight line Y.

Consider first the condition that its distance from AB = X.

The locus of a point at a distance X from AB consists of a pair of straight lines parallel to AB.

Next consider the condition that the distance from CD = Y.

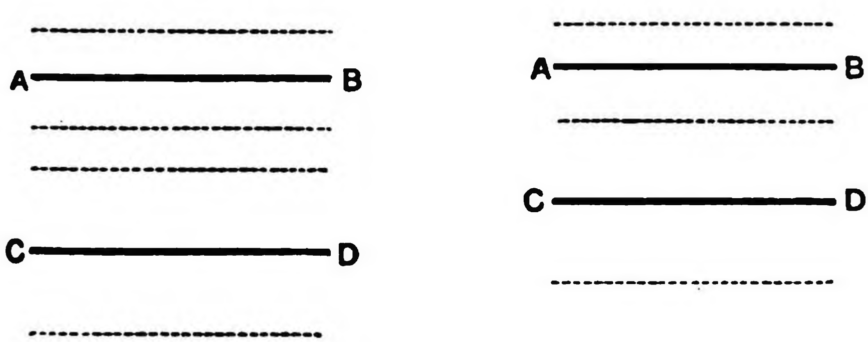
The locus of a point at a distance Y from CD consists of a pair of straight lines parallel to CD.

If AB is not parallel to CD the two loci will intersect in four points P, Q, R, S, and any one of these satisfies both the given conditions.

If AB is parallel to CD—

(1) There may be no point common to both loci, in which case the solution of the problem is impossible.

(2) One of the two lines forming one locus may coincide with one of the



two lines forming the other, in which case every point on that line satisfies both conditions, and the solution of the problem is indeterminate.

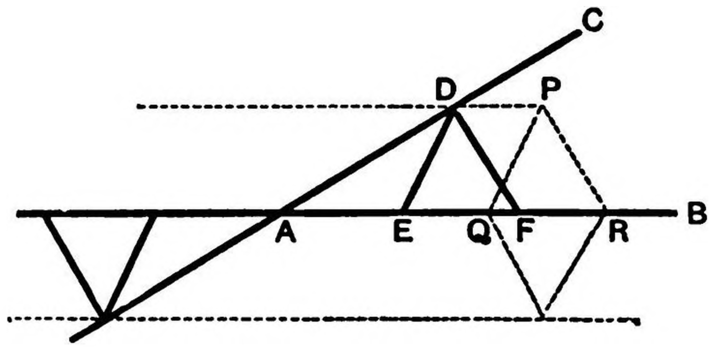
This will be the case when the sum of X *and* Y (*as in the diagram*) *or their difference is equal to the distance of* AB *from* CD.

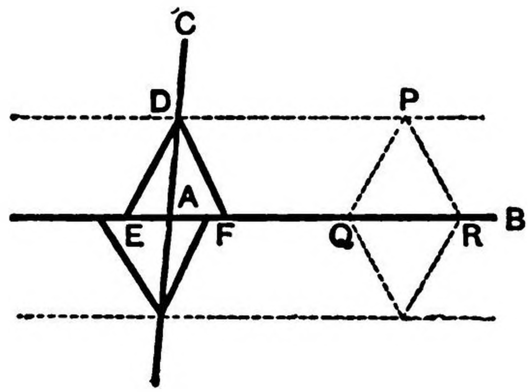
As a second example of the method take the problem—

AB and AC are two given straight lines. It is required to describe an equilateral triangle DEF, having its vertex D on AC, and its base EF of given length on AB.

When will E and F be on opposite side of A?

This last question implies that the lines may be produced through A.





Consider only the condition that the equilateral triangle has a base of given length on AB.

The locus of its vertex would evidently be a pair of lines parallel to AB. [I. 40.

But the vertex is also to lie on AC.

Therefore it must be at one of the intersections of AC with the pair of parallels.

Hence the solution.

Take QR anywhere on AB of the given length, and on it describe an equilateral triangle PQR. Through P draw PD parallel to AB, cutting AC in D. Through D draw DE, DF parallel to PQ, PR respectively.

$\angle$ DFE = int. and opp. $\angle$ PRQ,
= $\angle$ PQR, [I. 5.
= int. and opp. $\angle$ DEF.
$\angle$s of $\triangle$ DEF = $\angle$s of $\triangle$ PQR;
$\therefore$ DEF is equilateral.
and DE = PQ,
= QR.

The first equilateral triangle can be described on either side of QR, and there will be two solutions, as indicated in the diagram.

If E and F fall on the same side of A, one of the two angles made by the given straight lines is less than the interior angle of an equilateral triangle.

If E and F fall on opposite sides of A, each of these two angles is greater than the interior angle of an equilateral triangle.

Problems to be solved by the above method.

Ex. 114.—Construct a circle of given radius to pass through two given points.

Ex. 115.—Find a point which shall be at given distances from two given points.

To which of Euclid's Problems is this proposition equivalent?

Ex. 116.—Find a point at equal distances from two given points, and at a given distance from another given point.

Ex. 117.—Find a point at equal distances from two given straight lines, and at a given distance from another given straight line.

Ex. 118.—Find a point at equal distances from two given points, and at a given distance from a given straight line.

Ex. 119.—Find a point at equal distances from two given straight lines, and at a given distance from a given point.

Ex. 120.—Show how to draw a triangle ABC, having given the perpendicular AD, from A to BC, and the lengths of the two sides AB, AC. What condition limits the length of the given perpendicular? How many triangles can generally be drawn to fulfil the given conditions?

Ex. 121.—One vertex of an equilateral triangle is fixed, another is taken anywhere on a given straight line. Show that the locus of the third vertex is a pair of straight lines.

Hence describe an equilateral triangle with one vertex at a fixed point, and the other two one on each of two given straight lines.

Hence also, show that any number of equilateral triangles can be described with their vertices one on each of three given straight lines.

Ex. 122—O is a fixed point, and P any point on a given straight line; PO is produced to Q, so that OQ is equal to OP. Show that the locus of Q is a straight line parallel to the given one.

Hence find two points one on each of two given intersecting straight lines, such that the straight line joining them shall be bisected at a given point.

Ex. 123.—Find two points, one on each of two given circles, such that the straight line joining them shall pass through a given point and be bisected there.

II.—METHOD OF INTERSECTION OF SETS.

A second class of problems proposes, either directly or indirectly, *the determination of a straight line satisfying two conditions.* The method of solution is similar to the last.

Consider one of the given conditions only, and find if possible the set of lines satisfying it.

Next consider the other given condition by itself, and find, if possible, the set of lines satisfying it.

Any line which may happen to be common to both sets (and in which therefore the two sets, following the analogy of loci, may be said to *intersect*) *satisfies each of the given conditions.*

1. *If no line is common to the two sets* (in other words, if the two sets do not intersect) *there is no such line, and the solution of the proposed problem is impossible.*

2. *If more than one line is common to the two sets* (in other words, if the two sets intersect in more than one line), *there is more than one solution of the proposed problem.*

3. *If a pencil of lines is common to both sets, the solution is indeterminate.*

The *complete* solution of a problem includes an investigation (1) *Of the number of possible solutions,* and (2) *Of the particular conditions under which the problem may become impossible or indeterminate.*

As an instance of an application of the method, take the following problem:—

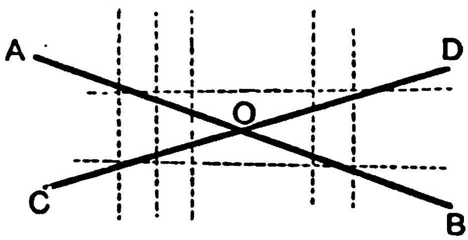
To draw a straight line which shall make equal angles with two given intersecting straight lines AOB, COD, *and be equidistant from two given points* X, Y.

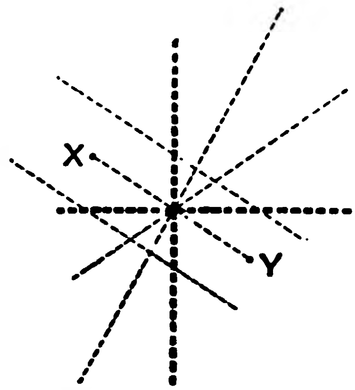
Consider first the condition that the angles made with AOB, COD are to be equal.

The set of straight lines making equal angles with AOB, COD consists

of two pencils of lines parallel respectively to the bisectors of the angles between them.

The set of lines equidistant from X and Y consists of straight lines of





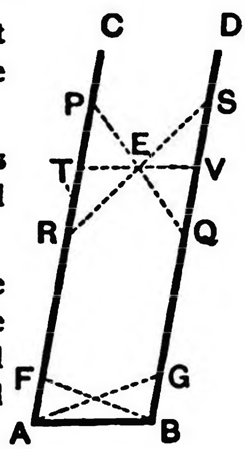
two pencils, one parallel to XY, the other of straight lines passing through the mid-point of XY.

Now lines drawn through the mid-point of XY parallel to the bisector of the angles AOC, AOD evidently belong to both sets, and satisfy both conditions: therefore the solution is always possible, and there are always two straight lines satisfying the given conditions.

Also if any one of the pencil of lines parallel to XY happens to be parallel to a bisector of either of the angles AOC, AOD, there will be a pencil (of parallel straight lines) common to both sets, and the problem will become indeterminate.

As a second example, take the problem :—

On a given base to construct a trapezoid equivalent to a given rectilineal figure, having an angle equal to a rectilineal angle, and having the difference of the two parallel sides equal to a given straight line.



Let AB be the given base, and X the given straight line, to which the difference of the parallel side is to be equal.

From its extremities draw two parallel straight lines AC, BD, one of them making with AB an angle equal to the given rectilineal angle.

The problem is reduced to drawing a straight line PQ from AC to BD, that shall make the figure APQB equivalent to a given rectilineal figure, and make the difference between AP and BQ equal to X.

First consider only the condition that the figure APQB is to be equivalent to a given rectilineal figure.

It is easy to show that if two straight lines PQ, RS cut off equal areas APQB, ARSB, they bisect each other, and conversely. Consequently all lines which satisfy the condition as to the area of the figure form a pencil passing through a certain fixed point E, which can be found by bisecting the side opposite to AB of a known parallelogram (I. 45, Cor.).

Next consider only the condition that the difference between AP, BQ is to equal X. From AC, BD cut off AF, BG, each equal to X.

It is easy to show that if the difference between AP and BQ is equal to X, PQ must be parallel to either AG or BF, and conversely. Consequently the set of lines satisfying the condition as to AP, BQ consists of two pencils of straight lines parallel respectively to AG, BF.

Now the two lines drawn through E parallel to AG and BF belong to both sets.

Therefore the solution is always possible, and there are always two trapezoids satisfying the required condition.

They will not be congruent unless the given rectilineal angle be a right angle.

III.—METHOD OF ANALYSIS AND SYNTHESIS.

This method consists chiefly in examining the properties of a figure in which the solution of the proposed problem is supposed to have been effected.

It may happen that the properties of the figure turn out on examination to be such that the student is able to effect its construction by Elementary Geometry. See page 4.

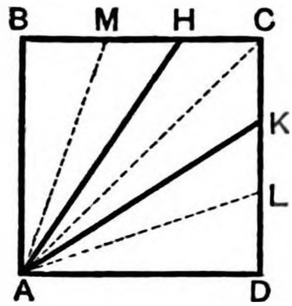
When we examine the properties of the figure, we are said to be performing **Geometrical Analysis.**

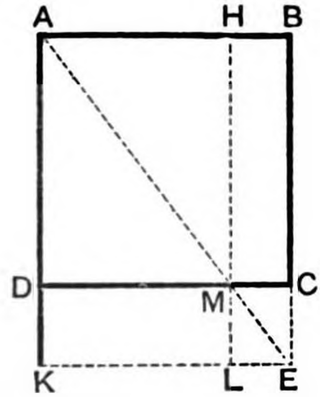
When we use the results of our investigation to construct the figure by Elementary Geometry, we are said to perform **Synthesis.**

As an example, take the following problem :—

To divide a square into three equivalent parts by two straight lines drawn through one angular point.

Let ABCD be the given square.





Analysis.—If we draw the diagonal AC, and thus divide the square into two equal parts, we see that one of the required lines AH must fall between AB and AC, and the other AK between AD and AC.

Also since $\triangle ACB = \triangle ACD$.
and $\triangle AHB = \triangle AKD$.
$\therefore \triangle AHC = \triangle ACK$.
Now $\triangle AHB =$ quadl. AHCK.
$=$ twice $\triangle AHC$.
$\triangle AHC = \frac{1}{3} \triangle ABC$.

Synthesis.—Divide BC into three equal parts, BM, MH, HC. [Ex. 111.

Divide CD into three equal parts, CK, KL, LD. Then AH, AK shall be the required straight lines.

For since $BM = MH = HC$.
$\therefore \triangle ABM = \triangle AMH = \triangle AHC$.
$\therefore$ each $= \frac{1}{3}$ of $\triangle ABC$.
$= \frac{1}{6}$ of sq. BD.
Similarly each of $\triangle$s ACK, AKL, ALD
$= \frac{1}{6}$ of sq. BD.

Now each of figures ABH, AHCK, AKD is made up of two of these equal $\triangle$s.

$\therefore$ each $= \frac{1}{3}$ of sq. BD.

As a second example of this method, take the problem :—

Having given one side of a rectangle equivalent to a given square, find the other side.

Let ABCD be the given square and AH one side of a rectangle equivalent to it.

Analysis.—Suppose AK taken along AD were equal to the other side of the rect., and the rect. AL with AH, AK for adjacent sides completed.

$\because$ rect. AL $=$ sq. AC.
$\therefore$ rect. DL $=$ rect. HC. [AX. 3.

This perhaps suggests to the student the figure of I. 43, *and the idea that* DL *and* HC *are complements of parallelograms about the diameter of a parallelogram.*

The diameter of this parallelogram would be along AM, which is known, and AB would be one of its sides. Hence the parallelogram can be constructed.

Synthesis.—Through H draw HM parallel to AD or BC, meeting CD in M.

Join AM, and let AM, BC (produced if necessary) meet in E.

Through E draw EK parallel to AB, meeting AD or AD produced in K. AK shall be the required side of the rectangle equivalent to AC.

For compt. DL = compt. HC. [I. 43.

∴ whole AL = whole AC. [Ax. 2.

∴ AK is required side.

Ex. 124.—To divide a triangle into three equal parts by straight lines drawn through a given point in one of its sides.

Ex. 125.—Draw a straight line DE parallel to the base BC of a triangle ABC, cutting AB in D and AC in E, so that DE shall be equal to the sum of BD and CE.

Ex. 126.—Find that straight line that would, if produced, bisect the angle between two given straight lines without producing the given straight lines to meet.

Ex. 127.—To divide a given straight line into two parts such that the square on one part shall be double of the square on the other part.

Remembering how to obtain a square double a given square (see note on Prop. 47) it is easy to draw a line divided in the way in which the given line is to be. Examine the whole figure and determine how one like it is to be drawn, *starting with the given straight line in the place of the one found.* The problem can be reduced to the construction of a triangle, having given the base and the adjacent angles. The student can deal in a similar way with the problem:—

Ex. 128.—To produce a given straight line so that the square on the whole line thus produced shall be double of the square on the produced part.

And also *remembering that* **the square on the altitude of an equilateral triangle is three times the square of half the base** with the problems:—

Ex. 129.—To divide a straight line into two parts such that the square on one is equal to three times the square on the other.

Ex. 130.—To produce a straight line so that the square on the whole line produced shall be equal to three times the square on the produced part.

MISCELLANEOUS EXERCISES.—II.

Ex. 131.—ABCD is a ||gm. To CD is applied another ||gm CDEF. To EF is applied another ||gm EFGH. Show that any ||gm on base AB, and between the ||s AB, HG, is equivalent to the figure ABCFGHED. Hence obtain another solution of I. 45.

Ex. 132.—Any △ can be divided into two isosceles △s and a kite.

Ex. 133.—From any point D in the base BC of an isosceles △ ABC straight lines are drawn || to the sides, and meeting them in E and F. Show that the sum of the two lines is the same for all positions of D.

Extend the theorem, DE and DF being any straight lines making the same given angle with the base (*e.g.* being perpr. to the equal sides).

Ex. 134.—Through any point P within an equilateral △ ABC a straight line pq is drawn || to BC and meeting AB, AC in p, q; rs, tv are similarly drawn || to CA, AB. Show that pq, rs, tv are together equal to two of the sides of ABC. Show also that if PD, PE, PF be drawn from P to the sides of ABC, and each having the same given inclination to those sides, their sum is independent of the position of P.

Ex. 135.—ABC is a triangle having AB equal to AC. D is any point on AB, and E a point on C such that AE = BD. Find the locus of the mid-point of DE.

Ex. 136.—Prove I. 8 by supposing the two triangles placed on opposite sides of a common base and having their vertices joined.

Ex. 137.—OX, OY are two fixed straight lines; along OX, OY are taken any two points A and B respectively, such that the sum of OA and OB is constant. Parallels to OY, OX through A and B meet in P. Show that P lies on a straight line equally inclined to OX, OY.

Ex. 138.—Construct a rhombus, having given one of its angles and the distance of its centre from each of its sides.

Ex. 139.—Construct a rectangle, having given one of its sides and the distance of its centre from each of its angles.

Ex. 140.—ABCD is a quadrilateral, having AB and DC || to each other, and together equal to BC. Show that the st. lines bisecting the angles B and C intersect on AD.

Ex. 141.—The sides of a △ subtend obtuse angles at the point within it equidistant from its sides.

Ex. 142.—Two ||gms have two diagonally opposite vertices in common Show that the remaining four vertices are vertices of a ||gm.

Ex. 143.—Make a square equivalent to three-fourths of a given square.

Ex. 144.—Two congruent △s can be placed in an infinite number of

ways, so that the figure common to both is a hexagon whose opposite sides are equal and parallel.

If each median of one of the two △s falls along that one of the other with which it coincides when one △ is applied to the other as in I. 4., show that the hexagon is equivalent to two-thirds of one of the △s.

Ex. 145.—Any △ can be divided into a kite and a triangle in three different ways.

Ex. 146.—A ‖gm can always be found equivalent to a given △, and having one side and one angle in common with it.

Hence any two equivalent △s on equal bases can be divided into congruent parts.

Ex. 147.—Find a point which is equidistant from the four sides of a kite.

Ex. 148.—Find a point which is equidistant from the four vertices of an axe-head.

Ex. 149.—Show how to find any number of points equidistant from a given point and a given straight line.

Ex. 150.—A number of right-angled △s have a common right angle and equal hypotenuses. Show that the mid-points of the hypotenuses all lie on the same circle.

Ex. 151.—To divide a given rectilineal angle into two parts of which one = one-seventh of the other.

Ex. 152.—If two regular figures are such that an exterior angle of one is equal to an interior angle of the other they must either be two squares or a triangle and a hexagon.

Ex. 153.—The sides AB, AC of a △ ABC are produced and the two exterior angles bisected. Show that one of the angles contained by the bisecting lines is equal to half the sum of the ∠s ABC, BCA.

Ex. 154.—From a point P, perprs. PD, PE, PF are drawn to the sides BC, CA, AB of a △ ABC; show that the squares on AF, BD, CE are together equal to the squares on AE, CD, BF. Prove also conversely that—

If points D, E, F be taken in the sides BC, CA, AB of a △ ABC, such that the sqs. on AF, BD, CE are together equal to the sqs. on AE, CD, BF, the perprs. to those sides at D, E, F are concurrent.

And show that two of the 'Properties of Triangles,' p. 99, may be demonstrated by the help of the second theorem.

Ex. 155.—In the figure I. 47 let perprs. MP, MQ be drawn from the point M, where AL cuts BC, to AB, AC, and produced to meet FG, HK in R, S. Show that the rectangles AR, AS are equivalent, and hence that FG, MA, KH, if produced, will all pass through one point.

Also prove the second theorem independently of the first.

Ex. 156.—On the sides AB, BC of a △ ABC any ‖gms ABFE, BCDL

are constructed, and EF, DL are produced to meet in O. On AC a ||gm ACHG is constructed, having AG, CH equal and || to OB. Prove that it is equivalent to the other two ||gms. (Legendre's *Éléments de Géométrie.*)

Ex. 157.—One vertex of an equilateral △ is at a given point, another is on a given circle; show that the locus of the third consists of two circles.

Hence show that any number of equilateral △s can be described, each with one vertex on each of three given circles.

Ex. 158.—O is a given point, P any point on a given circle; PO is produced to Q, so that OQ = OP. Show that the locus of Q is a circle.

Hence through a fixed point O to draw a straight line PQ which shall be bisected at O, and have one extremity on each of two given circles.

Ex. 159.—To describe an equilateral triangle having given the distances of a point from each of its vertices.

Ex. 160.—To construct a triangle having given its three medians.

Ex. 161.—To trisect a rectangle by lines drawn through the mid-point of one of its sides.

Ex. 162.—To trisect a ||gm by lines drawn through any given point in one of its sides.

Ex. 163.—OX, OY are two given straight lines, and P any point within the angle XOY. Find a straight line through P which shall make with the given straight lines the triangle of least possible area.

Ex. 164.—**Of all equivalent △s on the same base the isosceles will have the least perimeter.**

Ex. 165.—**Construct a △ having given two sides and the angle opposite one of them. When do the *data* afford two solutions? When is the solution impossible?**

Ex. 166.—**If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to two equal sides equal, the angles opposite the other equal sides are either equal or supplementary, and in the former case the triangles are congruent.**

Cor.—Hence the two triangles are congruent.

(1.) If the two angles given equal are right angles or obtuse angles.

(2.) If the angles opposite the other two equal sides are both acute, or both obtuse, or if one of them is a right angle.

(3.) If the side opposite the given angle in each triangle is not less than the other given side.

Ex. 167.—Construct a triangle having given the base, the vertical angle, and either the sum or difference of the other two sides.

Ex. 168.—Two quadrilaterals are equivalent when their diagonals are equal and intersect at the same angle. (Legendre's *Eléments de Géométrie.*

Ex. 169.—On the sides of a △ ABC are constructed the squares ABDE, ACGF, BCHI, and EF, GH, ID are joined; it is required to construct the △ ABC having given the lengths of these three st. lines. Reduce to Ex. by showing that each line is twice the length of a median of the △ ABC. (Vuibert's *Questions des Mathématiques Élémentaires.*)

Ex. 170.—Find a point in the interior of a given quadrilateral such that by joining it to the angular points, the quadrilateral is divided into four triangles, which are equivalent, two and two.

Through the mid-point of each diagonal draw a parallel to the other. The pt. required is the intersection of these parallels. (Vuibert's *Questions des Mathématiques Élémentaires.*)

Ex. 171.—On the diagonals of a ||gm, rectangles are constructed such that the sides opposite the diagonals intersect on a side of the ||gm. Show that the two rectangles are together equivalent to the ||gm. (Vuibert's *Questions des Mathématiques Élémentaires.*)

Ex. 172.—An *equilateral* hexagon has all its *sides* equidistant from a given point. Show that its alternate *angles* are equal, and that it has three *equal diameters* which are axes of symmetry.

Show how to construct such a figure, and that it need not be *equi-angular.*

Ex. 173.—An *equiangular* hexagon has all its *vertices* equidistant from a given point. Show that its alternate *sides* are equal, and that it has three equal *medians* which are axes of symmetry.

Show how to construct such a figure, and that it need not be *equilateral.*

Ex. 174.—Through the vertex A of a △ ABC a line is drawn || to the base BC. Show how to draw through B a line cutting AC in P and the above mentioned || in Q, so that BP shall be one-third of PQ.

Ex. 175.—If the opposite angles of a quadrilateral are supplementary, a point can be found which is equidistant from the four vertices.

Hence a circle can be described about such a quadrilateral.

Ex. 176.—If the sum of one pair of opposite sides of a convex quadrilateral is equal to that of the other two, a point can be found which is equidistant from the four sides.

A figure is called **convex** *when the join of any two points on its boundaries does not lie without the figure.*

The Corollaries to I. 32 are only true *as enunciated* for *convex polygons.*

For a discussion on polygons in general the student is referred to Henrici, *Congruent Figures,* Chap. VI. (Angles in Polygons).

THE HARPUR EUCLID.

BOOK II.

DEFINITION.

If a parallelogram has one angle a right angle it has all its angles right angles. [I. 46.

Such a parallelogram is called a rectangle.

It is said to be **contained by** any two of the straight lines which contain one of the right angles.

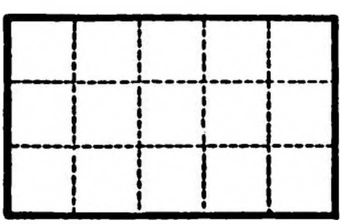
Thus a right angled ||gm ABCD *is spoken of as* 'the rectangle contained by AB and AD.'

The phrase 'rectangle contained by AB and AD' *is abbreviated into* 'rect. AB, AD,' or into 'rect. AB. AD.'

By the 'rectangle contained by two straight lines x and y' is meant the rectangle two of whose adjacent sides are equal to x and y respectively.

This way of describing a figure by means of two of its sides only is like that used by carpenters in speaking of a board '5 *inches by* 3.'

If two adjacent sides of a rectangle are equal to 5 *and* 3 *times the unit of length respectively it is easy to show that the rectangle is equal to* 15 *times the square on the unit of length. We have only to divide the two sides into* 5 *and* 3 *equal parts respectively, and draw through the points of section of each side lines parallel to the other.*



The whole rectangle is thus divided into 3 **rectangles**, *each of which is divided* **into 5 squares**, *and therefore contains* **3 times** 5 *squares.*

A similar division of the rectangle could obviously be made if the sides containing it were each a whole number of times the unit of length.

It would not be very difficult to extend the principle to cases where the lengths of the sides of the rectangle, in terms of the unit of length, were expressed one by a whole and one by a mixed number, or each by a mixed number.

Ex. 177.—*Draw a diagram showing that if the sides of a rectangle are* $2\frac{1}{2}$ *inches and* 3 *inches long the rectangle is equal to* $(2\frac{1}{2} \times 3)$ *square inches.*

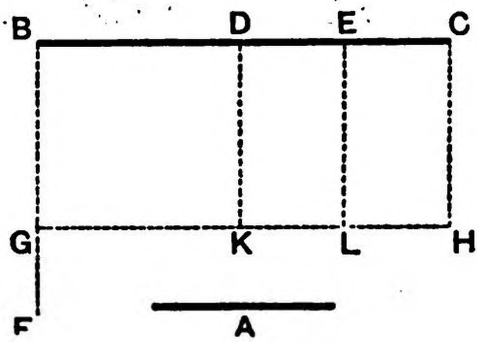
Ex. 178.—*Draw a diagram showing that if the sides of a rectangle are* $2\frac{1}{2}$ *inches and* $3\frac{1}{3}$ *inches the rectangle is equal to* $(2\frac{1}{2} \times 3\frac{1}{3})$ *square inches.*

We arrive, therefore, at the result that if a *and* b *are any whole or mixed numbers, a rectangle whose sides are* a *inches and* b *inches respectively is equal to* 'a b' *square inches.*

PROPOSITION 1. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

Let A and BC be two straight lines, of which BC is divided at D, E; then rect. A . BC=A . BD, A . DE, A . EC.



Draw BF ⊥r to BC, and cut off BG equal to A.

Through G draw a line ∥ to BC.

Through D, E, C draw DK, EL, CH ∥ to BG, meeting the ∥ through G in K, L, H.

Then all the figures are rectangles :— [I. 46, COR.

Fig. BH = figs. BK, DL, EH.

But BH = rect. A, BC for BG = A.

BK = rect. A, BD for BG = A.

DL = rect. A, DE for DK = BG = A.

EH = rect. A, EC for CH = BG = A.

Hence rect. A . BC = rects. A . BD, A . DE, A . EC.

NOTES.

The Syllabus enunciates the corresponding Proposition thus:—

The rectangle contained by two given lines is equal to the sum of the rectangles contained by one of these and the several parts into which the other is divided.

Numerical Illustration.—If A, BD, DE, EC *were* 5 *inches*, 6 *inches*, 2 *inches, and* 3 *inches long respectively*, BC *would be* 11 *inches.*

Rect. A, BC	*would contain*	55 *sq. in.*
Rect. A, BD	,,	30 ,,
Rect. A, DE	,,	10 ,,
Rect. A, EC	,,	15 ,,

Now $55=30+10+15$.

Algebraical Illustration.—If A, BD, DE, EC, *were* p *inches*, q *inches*, r *inches*, s *inches long respectively*, BC *would be* $q+r+s$ *inches long.*

Rect. A, BC	*would contain*	$p(q+r+s)$ *sq. in.*
Rect. A, BD	,,	pq *sq. in.*
Rect. A, DE	,,	pr ,,
Rect. A, EC	,,	ps ,,

Now by Algebra $p(q+r+s)=pq+pr+ps$.

Ex. 179.—**Show** (*by two applications of* II. 1) **that the square on a st. line is equal to four times the square on half of it.**

(*Take the undivided line equal to half the divided one.*)

Give numerical and algebraical illustrations of the above theorem.

PROPOSITION 2.

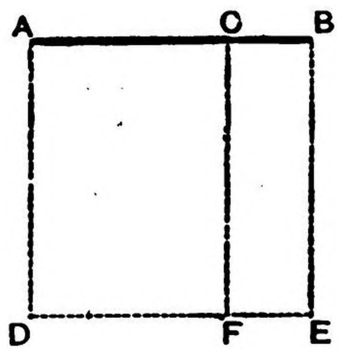
If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let AB be divided at C.

The rect. AB, AC with rect. AB, BC = sq. on AB.

On AB describe the sq. ADEB.

Through C draw CF ∥ to AD.



Figs. AF, CE = fig. AE.

But AF = rect. AB, AC for AD = AB,

CE = rect. AB, BC for BE = AB,

and AE is the sq. on AB.

NOTE.

This is a mere Corollary to II. 1, the *undivided line* being equal to the *divided line*, and the latter being divided into *two* parts.

Numerical Illustration.—A st. line AB, 5 *inches long, might be divided into two parts,* AC 3 *inches long,* CB 2 *inches long.*

Sq. on AB	*would contain*	25	*sq. in.*
Rect. AB, AC	"	15	"
Rect. AB, BC	"	10	"

Now $25 = 15 + 10$.

Algebraical Illustration.—Let the whole line AB *be* p, *and the parts,* AC, CB, q *and* r *inches long respectively.*

Then $p = q + r$.

Now area of sq. on AB (*in sq. inches*)$= p^2$,

$= p(q + r)$,

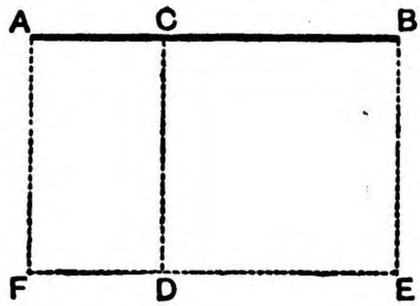
$= pq + pr =$ *area of rect.* AB, AC + *area of rect.* AB, BC.

PROPOSITION 3.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let AB be divided at C.

Then rect. AB, BC=rect. AC, CB, with sq. on BC.



On BC describe the square CDEB.

Through A draw AF || to CD,
meeting ED produced in F.

Fig. AE=figs. AD, CE.

But AE=rect. AB, BC for BE=BC,
AD=rect. AC, CB for CD=BC,
and CE is the sq. on BC.

∴ rect. AB, BC=rect. AC, CB with sq. on BC.

NOTE.

This is a mere Corollary to II. 1, the *undivided* line being equal to *one of the two parts into which the other is divided.*

Numerical Illustration.—A line AB, *7 inches long, might be divided into two parts*—AC, *three inches long, and* CB, *4 inches long.*

Rect. AB, BC *would contain* 28 *sq. in.*
Rect. AC, CB ,, 12 ,,
Sq. on BC ,, 16 ,,
Now $28 = 12 + 16$.

Algebraical Illustration.—Let the whole line AB *be* p, *and the parts* AC, CB *be* q *and* r *inches long respectively.*

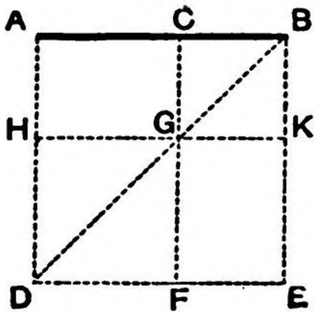
Then $p = q + r$.

Now area of rect. AB, BC (*in sq. in.*) $= pr$,
$= (q+r)r$,
$= qr + r^2$,
$=$ *area of rect.* AC, CB $+$ *area of sq. on* CB.

PROPOSITION 4. THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

Let AB be divided in C; then sq. on AB = sqs. on AC, CB, with twice rect. AC, CB.



On AB describe a sq. ADEB.

Join BD. Through C draw CGF ∥ to AD or BE, cutting BD in G; through G draw HGK ∥ to AB or DE.

∵ CF is ∥ to AD,

∴ ext. ∠CGB = int. and opp. ∠ADB, [I. 29.

= ∠ABD (∵ AD = AB), [I. 5.

∴ CG = CB. [I. 6.

But CB = GK, and CG = BK, [I. 34.

∴ CBKG is equilateral.

And ∵ it is a ∥gm, and has one ∠CBK a rt. ∠,

∴ all its ∠s are rt. ∠s. [I. 46, COR.

∴ it is a square.

Similarly HF is a square, and = sq. on AC (∵ HG = AC). [I. 34.

Now compt. AG = compt. GE.

∴ AG, GE together = twice AG.

= twice rect. AC, CB (∵ CG = CB).

Add CK, HF which = sqs. on AC, CB.

∴ AG, GE, CK, HF = sqs. on AC, CB, with twice rect. AC, CB.

i.e. AE = sqs. on AC, CB, with twice rect. AC, CB.

i.e. sq. on AB = sqs. on AC, CB, with twice rect. AC, CB.

COROLLARY.—**Each of the parallelograms about the diameter of a square is a square.**

NOTES.

We might have divided the Corollary into two distinct propositions, which have been separately demonstrated—

(i.) *Each of the parallelograms about the diameter of a* **rhombus** is a **rhombus.**

(ii.) *Each of the parallelograms about the diameter of a* **rectangle** is a **rectangle.**

Numerical Illustration.—A st. line AB, 11 *inches long, might be divided into two parts*—AC, 5 *inches long, and* CB, 6 *inches long.*

Sq. on AB	*would contain*	121	*sq. in.*
Sq. on AC	„	25	„
Sq. on CB	„	36	„
Rect. AC, CB	„	30	„

Now $121 = 25 + 36 + 60.$

Algebraical Illustration.—Let the whole st. line AB *be* z, *and let the parts* AC, CB *be* x *and* y *inches long respectively,*

Then area of square on AB *in sq. inches* $= z^2$,
$= (x+y)^2$,
$= x^2 + y^2 + 2xy$,
$=$ *Area of sqs. on* AC, CB, $+$ *twice area of rect.* AC, CB.

Ex. 180.—If a ||gm is not equilateral, the ||gms about its diameter are not equilateral.

Ex. 181.—If a ||gm is not rectangular, the ||gms about its diameters are not rectangular.

Ex. 182.—Prove by II. 4 that the square on a straight line is equal to four times the square on half of it.

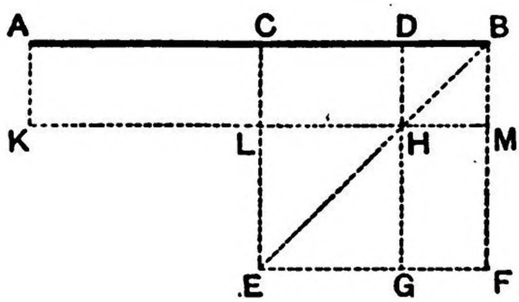
Ex. 183.—A, B, C, D are points in a st. line such that AB = BC = CD. Show that sq. on AD = sum of sqs. on AC, BD, and BC.

Ex. 184.—In the figure R and M are taken on DE, EB, so that DR and EM each = BC. Show that HRMC is a square equivalent to sqs. on AC, CB.

PROPOSITION 5.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let AB be bisected at C and divided unequally at D; then rect. AD, DB with sq. on CD=sq. on CB.



On CB describe the square CEFB.

Join BE; through D draw DHG ‖ to BF, cutting BE in H; through H draw MHLK ‖ to AB; through A draw AK ‖ to CE.

Compt. CH=compt. HF,

∴ CM=DF;

but AL=CM (∵ AC=CB), [I. 36.

∴ AL=DF,

∴ AH=DF, CH.

But AH=rect. AD, DB (∵ DH=DB),

∴ rect. AD, DB=DF, CH.

But sq. on CD=LG;

∴ rect. AD, DB with sq. on CD=DF, CH, LG,

=CF,

=sq. on CB.

COROLLARY.—**The difference of the squares on two unequal straight lines (AC, CD) is equal to the rectangle contained by their sum and difference** (AD and DB).

Numerical Illustration.—A straight line AB 8 *inches long might be divided into unequal parts* AD, DB 5 *and* 3 *inches long, as well as into two equal parts* AC, CB *each* 4 *inches long.*

Then rect. AD, DB *would contain* 15 *sq. in.,*
sq. on CD ,, 1 ,,
and sq. on CB ,, 16 ,,
Now $15+1=16$.

Algebraical Illustration.—Let AC, CB *each be* x *inches and* CD y *inches long. Then* AD *will be* $x+y$ *and* DB $x-y$ *inches long.*

$$\textit{Now } (x+y)(x-y)+y^2=x^2-y^2+y^2,$$
$$=x^2.$$

$\therefore$ *No. of sq. in. in rect.* AD, DB *and sq. on* CD$=$*No. of sq. in. in sq. on* CB.

Ex. 185.—**To find a point in a given straight line such that the rectangle contained by the two parts into which it divides the line shall be the greatest possible.**

Ex. 186.—**Of all rectangles of the same perimeter the square has the greatest area.**

Ex. 187.—**The difference of the squares on AD, DB is equal to twice the rectangle contained by CD, AB.** (Prove by II. 5, Cor.)

Note.—The following is also a Corollary to II. 5:—

Let AB be bisected in C and produced to D; then rect. AD, DB with sq. on CB$=$sq. on CD.

Produce CA to E, making AE equal to BD.

Then CE$=$CD,
and EB$=$AD.
Rect. EB, BD$+$sq. on CB$=$sq. on CD. [II. 5.

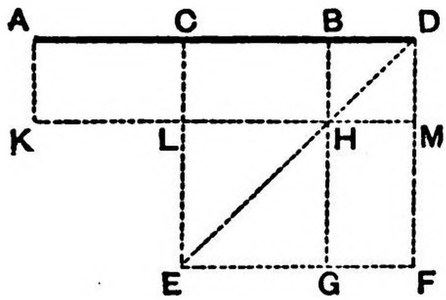
$\therefore$ rect. AD, DB$+$sq. on CB$=$sq. on CD ($\because$ EB$=$AD).

This may be taken as *alternative proof* of II. 6.

PROPOSITION 6.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let AB be bisected at C and produced to D;



then rect. AD, DB with sq. on CB=sq. on CD.

On CD describe the square CEFD. Join DE.

Through B draw BHG ‖ to DF cutting DE in H. Through H draw KLHM ‖ to AD. Through A draw AK ‖ to CE.

AL=CH (∵ AC=CB)

=compt. HF.

To each add CM,

∴ AM=CM, HF;

But AM=rect. AD, DB (∵ DM=DB),

∴ rect. AD, DB=CM, HF;

to each add LG, which=sq. on CB.

∴ rect. AD, DB with sq. on CB=CM, HF, LG.

=CF.

=sq. on CD.

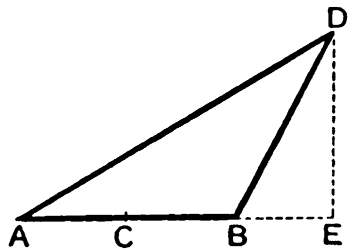
Ex. 188.—**Give a numerical illustration of the truth of II. 6.**

Ex. 189.—**Give an algebraical illustration of the truth of II. 6. (CD=x; AC=CB=y.)**

Ex. 190.—Show that assuming II. 6, a proof might be given of II. 5, like the alternative proof given for II. 6.

Ex. 191.—Deduce Corollary to II. 5 from II. 6.

Ex. 192.—**The difference of the squares on AD, DB=twice the rectangle contained by CD, AB.** (See Ex. 187.)



Ex. 193.—**AB is a straight line bisected in C. From any point D a perpendicular DE is drawn to AB. Show that the difference of the squares on AD, DB is equal to twice the rectangle contained by CE and AB.**

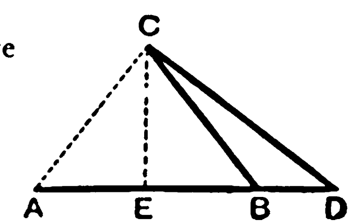
Sq. on AD=sqs. on AE, ED,
and sq. on BD=sqs. on BE, ED.
∴ diff. of sqs. on AD, BD=diff. of sqs. on AE, BE,
= twice rectangle CE, AB.

Ex. 194.—Hence—**The locus of a point such that the difference of the squares of its distances from two given points is constant is a straight line perpendicular to their join.**

What particular case of this general proposition has been demonstrated previously?

Ex. 195.—**C is** *any point* **equidistant from two fixed points A and B; D any point in AB or AB produced; then the rectangle AD, DB is equal to the difference of the sqs. on CB, CD.**

(*Generalisation of* II. 5 *and* 6.)



Draw CE perpendicular to AB, and prove that AE=EB.

Sq. on CB=sqs. on CE, EB.
Sq. on CD=sqs. on CE, ED.

∴ diff. of sqs. on CB, CD=diff of sqs. on ED, EB.
=rect. AD, DB. II. 5 and 6.

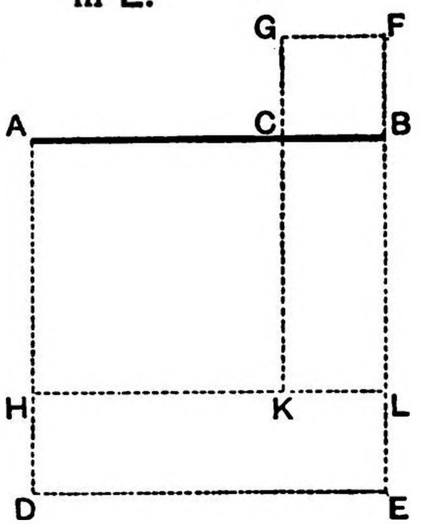
Hence—**If D is a fixed point within or without a given circle, and any straight line passing through D cut the circle in A and B, the rectangle AD, DB, is of the same magnitude however the line be drawn.** (See pp. 227, 228.)

PROPOSITION 7.

If a straight line is divided into two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part together with the square on the other part.

Let AB be divided at C. Then the sqs. on AB, BC = twice rect. AB, BC with sq. on AC.

On AB describe the square ADEB; on AC, on the same side, the square AHKC, and on BC on the opposite side the square BFGC, and produce HK to meet EB in L.



∵ CAH and CAD are rt. ∠s,
∴ AH falls along AD.
∵ ACK is a rt. ∠,
∴ BCK is a rt. ∠;
but BCG is also a rt. ∠,
∴ CG is in the same straight line with CK.
∵ ABF and ABE are rt. ∠s,
∴ BF is in the same straight line with BE.
∵ AD = AB,
and AH = AC,
∴ HD = BC
but DE = AB,
∴ HE = rect. AB, BC.

Again, ∵ CK = AC,
and CG = CB,
∴ GK = AB;
but GF = BC,
∴ GL = rect. AB, BC.

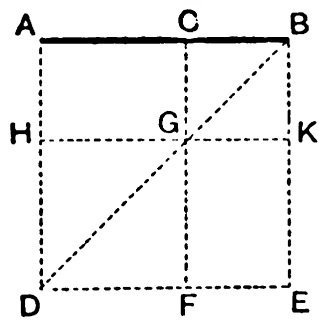
Now the figure ADEFGC is made up of the two squares AE, BG;
it is also made up of the figures HE, GL, AK.

∴ AE, BG = HE, GL, AK.

∴ Sqs. on AB, BC = twice rect. AB, BC, together with the square on AC.

(*This demonstration is taken from that of the corresponding Proposition in the 'Elements of Geometry,' by the Association for the Improvement of Geometrical Teaching. A similar one is given in Leslie's 'Elements.'*

Alternative Proof.—Let AB be divided in C; then sqs. on AB, BC=2 rect. AB, BC with sq. on AC.



On AB describe the square ADEB.

Join BD: through C draw CGF ‖ to AD, cutting BD in G; through G draw HGK ‖ to AB.

Compt. AG=compt. GE.

∴ AK=CE,

∴ AK, CE=2 AK,

=2 rect. AB, BC (∵ BC=BK).

But HF=sq. on AC.

∴ AK, CE, HF=2 rect. AB, BC with sq. on AC.

i.e. AK, GE, HF, CK=2 rect. AB, BC with sq. on AC.

i.e. AE, CK=2 rect. AB, BC with sq. on AC.

i.e. sqs. on AB, BC=2 rect. AB, BC with sq. on AC.

Ex. 196.—The student should also learn the enunciation in the subjoined form:—

The square on the difference of two lines is less than the sum of the squares of those lines by twice the rectangle contained by them. (Syllabus.)

Ex. 197.—Show that the Algebraical formula

$$(x-y)^2=x^2+y^2-2xy$$

is an illustration of II. 7.

Ex. 198.—Show that a demonstration can be given of II. 4 identical in method with that of the first of II. 7.

PROPOSITION 8. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

Let the st. line AB be divided into any two parts at the point C; four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the line made up of AB and BC.

Produce AB to D, so that $BD = BC$.

Then AD is made up of AB and BC.

Now sq. on $AD =$ sqs. on AB, $BD +$ twice rect. AB, BD, [II. 4.

$=$ sqs. on AB, $BC +$ twice rect. AB, BC ($\because BD = BC$),

$=$ twice rect. AB, $BC +$ sq. on $AC +$ twice rect. AB, BC, [II. 7.

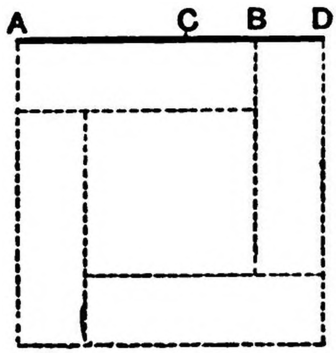
$=$ four times rect. AB, $BC +$ sq. on AC.

NOTE.

In examinations in which the use of the symbol + is not admitted, it may be avoided by a judicious use of the word *with*, as in the preceding Propositions.

NOTES.

1. The truth of II. 8 may also be easily demonstrated by drawing the sq. on AD, and dividing in the way indicated in the annexed diagram.



The demonstration is left as an exercise to the student.

2. This proposition might be enunciated thus :—

The square on the sum of two straight lines exceeds the square on their difference by four times the rectangle contained by them.

Ex. 199.—Give an arithmetical illustration of the truth of II. 8.

Ex. 200.—Show that the Algebraical Identity

$$(x+y)^2=(x-y)^2+4xy.$$

illustrates the truth of II. 8.

Ex. 201.—Prove II. 8 by Cor. to II. 5.

Ex. 202.—If a rectangle is equivalent to a given square its perimeter is least when it is congruent with the square.

Ex. 202 (a).—If a st. line be divided into 5 equal parts, show by II. 8 that the square on the whole line is equal to the sum of the squares on the lines made up respectively of 3 and of 4 of the equal parts (Cresswell's *Supplement to Euclid.*)

PROPOSITION 9. THEOREM.

If a straight line be divided into two equal and into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts at C, and into unequal parts at D, then the squares on AD, DB shall be together double of the squares on AC, CD.

Sq. on AD=sqs. on AC, CD+twice rect. AC, CD, [II. 4.
=sqs. on AC, CD+twice rect. CB, CD
(∵ AC=CB).

∴ Sqs. on AD, DB=sqs. on AC, CD+twice rect. CB, CD
+sq. on DB.
=sqs. on AC, CD+sqs. on CB, CD, [II. 7.
=twice sq. on AC+twice sq. on CD
(∵ AC=CB).

PROPOSITION 10. THEOREM.

If a straight line be bisected and produced to any point, the squares on the whole line thus produced and the part produced are together double of the square on half the line and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected in C and produced to D; the squares on AD, DB shall be together double of the squares on AC, CD.

Sq. on AD = sqs. on AC, CD + twice rect. AC, CD, [II. 4.
= sqs. on AC, CD + twice rect. CB, CD
($\because$ AC = CB);

$\therefore$ Sqs. on AD, DB = sqs. on AC, CD + twice rect. CB, CD + sq. on DB.
= sqs. on AC, CD + sqs. on CB, CD, [II. 7.
= twice sq. on AC + twice sq. on CD
($\because$ AC = CB).

NOTES.

1. Observe the identity of the demonstration of II. 9 and II. 10, and the close agreement of their particular enunciations.

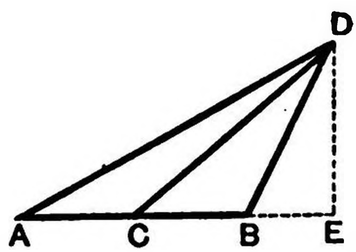
2. II. 9 and II. 10 could be enunciated thus :—

The squares on the sum and difference of two given straight lines are together double of the squares on the two given straight lines.

Generalisation of II. 9 and II. 10 :—

Ex. 203.—**C is the mid-point of AB, and D any other point; the squares on AD, DB are together double of the squares on AC, CD.**

Draw DE ⊥r to AB, or AB produced.



Sqs. on AD, DB = sqs. on AE, ED + sqs. on BE, ED, [I. 47.
= sqs. on AE, EB + twice sq. on ED,
= twice sq. on AC + twice sq. on CE + twice sq. on ED, [II. 9 and 10.
= twice sq. on AC + twice sqs. on CE, ED,
= twice sq. on AC + twice sq. on CD. [I. 47.

The figure only illustrates the case in which E *is in* AB *produced. The student should draw the other possible figures, and satisfy himself that the theorem is true universally.*

Ex. 204.—**The sum of the sqs. of the distances of a point (D) from two given fixed pts. (A and B) is constant. Show that the locus of D is a circle whose centre is the mid-pt. (C) of their join (AB).**

Ex. 205.—Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on its medians.

Ex. 206.—The sq. on the distance of a point (D) from a given point (A) is double the sq. of its distance from a given point (C). Show that the locus of D is a circle.

Produce AC *to* B *so that* CB=AC, *and use* Ex. 203.

Ex. 206 (a).—ABCD is a rectangle, P any point. Show that sqs. on PA, PC=sqs. on PB, PD. Show also that, if PE, PF, PG, PH are drawn ⊥ to AB, BC, CD, DA the sqs. on PA, PB, PC, PD are together double of the sqs. on PE, PF, PG, PH.

Ex. 206 (b).—**The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.** Gregory St. Vincent's *de Quadratura Circuli.* *Use* Exx. 63, 182, 203.

This is a particular case of a more general theorem published by Euler in the St. Petersburg Memoirs and given as Ex. 509.

Ex. 206 (c).—The diagls. AC, BD of a ▱gm. ABCD cross at E: P is any other pt. Show that the sum of the sqs. on PA, PB, PC, PD is equal to four times the sq. on EP together with the sqs. on AB, BC. *Use* Ex. 203.

Ex. 206 (d).—In any quadl. the sqs. on the diagls. are together equal to twice the sum of the sqs. on the st. lines joining the mid. pts. of opposite sides.

Use Exx. 63, 69 (*a*), 206 (*b*).

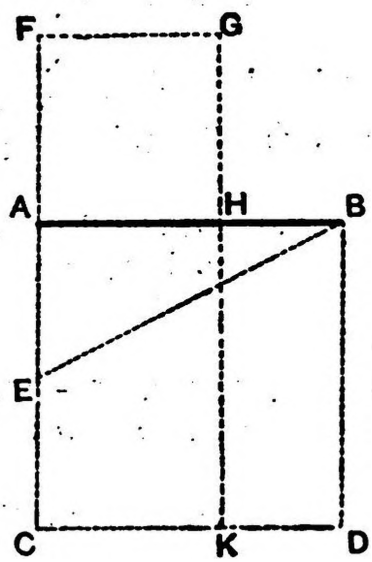
Ex. 206 (e).—In a quadl. ABCD, AC=CD, AD=BO, and the angles ACB, ADC are supplementary; shew that sq. on AB=sum of sqs. on BC, CD, DA.

Produce AD *to* E *so that* AD=DE: *prove that* CE=AB, *and use Ex.* 203.

PROPOSITION 11. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one part shall be equal to the square on the other.

Let AB be the given st. line; it is required to divide AB into two parts such that the rectangle contained by AB and one of its parts shall be equal to the square on the other.



On AB describe the square ABDC.

Bisect AC in E.

Join BE and produce CA to F, making EF equal to EB.

On AF describe the square AFGH. Then AB shall be divided at H, so that rect. AB, BH = sq. on AH.

Produce GH to meet CD in K.

Now rect. CF, FA with sq. on EA = sq. on EF, [II. 6.
= sq. on EB ($\because$ EF = EB),
= sqs. on EA, AB; [I. 47.
$\therefore$ rect. CF, FA = sq. on AB,
= AD.

But FK = rect. CF, FA ($\because$ FG = FA),
$\therefore$ FK = AD.
$\therefore$ FH = HD.
i.e., sq. on AH = rect. BD, BH.
= rect. AB, BH ($\because$ BD = AB).

NOTE.—*The above construction might be obtained by the solution of a quadratic equation. For let* a = *length of* AB, and x = *regd. length of* AH;

$$\textit{then } x^2 = a\,(a - x),$$
$$= a^2 - ax,$$
$$\therefore x^2 + ax = a^2.$$

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = \frac{5}{4}a^2,$$

$$\therefore x + \frac{a}{2} = \pm \frac{a}{2}\sqrt{5}.$$

$$x = \pm \frac{a}{2}\sqrt{5} - \frac{a}{2}.$$

Selecting the positive root it is easy to see that the line EB *is* $\frac{a}{2}\sqrt{5}$ (see p. 90). AH *must be equal to the difference of* EB *and* EA, *and* $\therefore$ *to* AF.

NOTES.

A straight line AB divided at H so that the rectangle AB, BH is equal to the square on AH is said to be divided '**in medial section**' or to be '**medially divided.**'

Note that CF is medially divided at A.

The proposition is not used by Euclid until IV. 10. The inscription of a regular pentagon in a given ⊙ will be found to depend on it. It may be useful for the student of Practical Geometry to note that—

CF = diagonal of regular pentagon on base AB.
AH = side of regular pentagon with diagonal AB.
AH = side of regular decagon in ⊙ with radius AB.
CH = BF = side of regular pentagon in ⊙ with radius AB.

A recollection of the construction of II. 11 will then enable him—

(1) *To describe a regular pentagon on a given straight line.*

(2) *To describe a regular pentagon with a given diagonal.*

(3) *To inscribe a regular decagon in a given* ⊙.

(4) *To inscribe a regular pentagon in a given* ⊙.

For the *proof* of these assertions he is referred to IV. 10, 11.

Ex. 207.—**To produce a given st. line (CA) to a point (F) such that the rectangle contained by the whole st. line thus produced (CF), and the part produced (AF), may be equal to the square on the given st. line (CA).**

Ex. 208.—**To produce a given st. line (BA) to a point (H) such that the rectangle contained by the whole st. line thus produced (BH), and the given st. line (AB), may be equal to the square on the part produced (AH).** (Syllabus.)

Obtain EB *as in* II. 11, *and produce* EC *to* F, *making* EF = EB ; *then follow the construction and demonstration of* II. 11 *as closely as possible.*

Ex. 209.—Investigate the solution of Ex. 207 by means of an algebraical equation.

Ex. 210.—Investigate the solution of Ex. 208 by means of an algebraical equation.

Ex. 211.—Prove that if, in the figure of II. 11, CH be produced to cut FB it will cut it at right angles. Enunciate and prove a similar theorem as regards the figure of Ex. 208. Show also that FH, CB will cut at right angles if produced.

Ex. 211 (a).—In the fig. of II. 11, if FG, CH, DB be produced they will meet in a point. *Use* Ex. 91.

Ex. 211 (b).—In the fig. of II. 11, the sqs. on CF, FB=4 times sq. on AB. *Use* I. 47 *and* II. 10.

Since the segments CA, AF of the st. line CF medially divided at A are equal respectively to the given st. line AB and its greater segment AH, it is clear that this peculiar division of any st. line being once obtained, a series of other st. lines all possessing the same property may readily be found by repeated additions. Thus let AB be so cut that sq. on BC= rect. AB, AC. Take D, E, F, G, etc., successively, along AB produced, such that BD=BA, DE=DC, EF=EB, FG=FD, and so on.

Then CD, BE, DF, EG, etc., are medially divided at B, D, E, F, etc. Even if the section of AB were at first assumed inexact, the series of combinations would always approach to greater accuracy. Assuming the segments AC, CB of AB as at first equal and each denoted by 1, the following successive numbers will result from a continued summation—

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc.

a kind of approximation first noticed in this actual case by Girard.

Note that $144 \times 55 = 89^2 - 1$; $89 \times 34 = 55^2 + 1$.

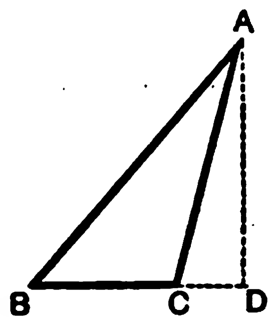
(Condensed from Leslie's *Elements* ; see also *English Mechanic*, vol. lii. p. 186).

PROPOSITION 12.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by the side on which, when produced, the perpendicular falls and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let ABC be a △ having the ∠ ACB obtuse.

From A draw AD ⊥r to BC produced.



Then sq. on AB > sqs. on AC, CB by twice rect. BC, CD.

BD is divided at C

∴ sq. on BD=sqs. on BC, CD, with twice rect. BC, CD. [II. 4.

To each add sq. on DA.

∴ sqs. on BD, DA=sqs. on BC, CD, DA, with twice rect. BC, CD.

i.e. sq. on BA=sqs. on BC, CA, with twice rect. BC, CD (∵ ∠ D is right).

i.e. sq. on BA > sqs. on BC, CA by twice rect. BC, CD.

NOTES.

N.B.—The demonstration might be abbreviated thus—

sq. on BA = sqs. on BD, DA, [I. 47.
= sqs. on BC, CD, DA, with twice rect. BC, CD, [II 4.
= sqs. on BC, CA, with twice rect. BC, CD, [I. 47.
i.e. sq. on BA > sqs. on BC, CA by twice rect. BC, CD.

The point where the perpendicular through a given point to a given straight line meets that line is called the 'Projection' of the given point on the given straight line.

Thus, in the figure of I. 12, point H is the projection of the given point C on the given straight line AB.

If the given point is on the given straight line it coincides with its projection upon it.

The join of the projections of two given points upon a given straight line is called the projection of the join of the two given points.

Thus, in the figure of I. 47,

DL	is the projection of	AB on DE.
DL	,,	AD on DE.
LE	,,	AC on DE.
FG	,,	BC on FG.
AB	,,	CF on HB.

With the use of the word projection *in the above sense the enunciation of* II. 12 *can be expressed in a more easily remembered form.*

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing it by twice the rectangle contained by either of these sides and the projection of the other upon it.

Ex. 212.—Any straight line is equal to its projection upon a parallel straight line.

Ex. 213.—**Equal straight lines have equal projections upon any straight line which is equally inclined to them.**

Ex. 214.—**Equal straight lines which have equal projections upon the same straight line must be equally inclined to it.**

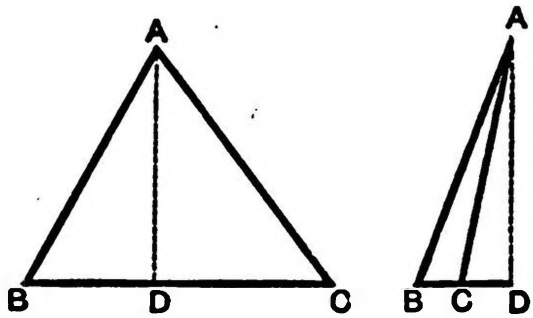
Ex. 215.—The rectangle BC, CD is equal to the rectangle contained by AC and the projection of BC on AC, in the fig. of II. 12.

Ex. 216.—Straight lines which have equal projections upon a straight line to which they are equally inclined are equal.

PROPOSITION 13.

In every triangle the square on the side subtending an acute angle is less than the squares on the sides containing that angle by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

Let ABC be any $\triangle$ having $\angle$ ABC acute.



(i.) If AC is not $\perp$r to BC, from A drop a $\perp$r AD on BC, or BC produced.

Then sq. on AC < sqs. on CB, BA by twice rect. CB, BD.

Now either BC is divided at D, or BD at C.

$\therefore$ in both cases sqs. on CB, BD = twice rect. CB, BD with sq. on DC, [II. 7.

to each add sq. on DA.

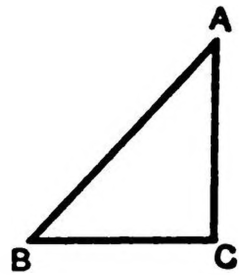
$\therefore$ sqs. on CB, BD, DA = twice rect. CB, BD with sqs. on DC, DA.

$\therefore$ sqs. on CB, BA = twice rect. CB, BD, with sq. on CA ($\because$ $\angle$s at D are right);

i.e. sq. on AC < sqs. on CB, BA by twice rect. CB, BD.

(ii.) Next let AC be ⊥r to BC.

Then sq. on AC < sqs. on CB, BA by twice sq. on BC.



Sq. on AB=sqs. on BC, CA (∵ ∠ ACB is right).

To each add sq. on BC.

Then sqs. on AB, BC=twice sq. on BC, with sq. on AC.

∴ sq. on AC < sqs. on CB, BA by twice sq. on BC.

NOTES.

N.B.—The above demonstrations might be abbreviated thus—

(i.) Sqs. on CB, BA=sqs. on CB, BD, DA, [I. 47.

=twice rect. CB, BD with sqs. on CD, DA, [II. 7.

=twice rect. CB, BD with sq. on CA, [I. 47.

i.e. Sq. on CA < sqs. on CB, BA by twice rect. CB, BD.

(ii.) Sq. on BA= sqs. on CB, CA, [I. 47.

∴ Sqs. on CB, BA=twice sq. on CB with sq. on CA.

i.e. Sq. on CA < sqs. on CB, BA by twice sq. on BC.

The enunciation should also be learned in the subjoined form.

In every triangle the square on the side subtending an acute angle is less than the squares on the sides containing that angle by twice the rectangle contained by either of these sides and the projection of the other upon it.

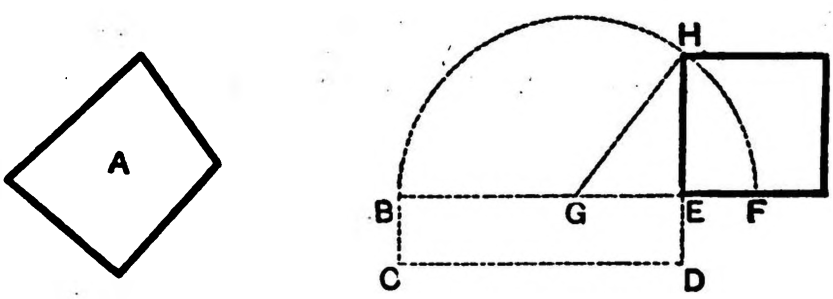
PROPOSITION 14.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectl. fig. It is required to make a sq. = A.

Make the rectangle BCDE = A. [I. 45.

Then if BE = ED the problem is solved; but, if not, produce BE to F so that EF = ED.



Bisect BF in G; with centre G, radius GF, describe a semicircle BHF.

From E draw EH $\perp$r to BF.

Join GH. The sq. on EH shall be equal to A.

BF is bisected at G and divided unequally at E.

$\therefore$ rect. BE, EF with sq. on EG = sq. on GF,
= sq. on GH ($\because$ GH = GF),
= sqs. on EG, EH
($\because$ $\angle$ GEH is right).

Take away the sq. on EG.

$\therefore$ rect. BE, EF = sq. on EH.

i.e. BD = sq. on EH.

$\therefore$ fig. A = sq. on EH.

NOTES.

1. The greater part of the Proposition is devoted to showing how to construct a **square** equivalent to a given **rectangle**.

Note that, in order to construct a **square** *equivalent to the* **rectangle contained by two given straight lines,** *it is by no means necessary to construct the rectangle itself. We have only to* **produce one of the given lines (BE) to F, making the produced part (EF) equal to the other,** *and proceed as in the Proposition.*

2. *In many cases it is easy to find the length and breadth of a rectangle equivalent to a given rectilineal figure by a special method.* See p. 85.

If no special method is available, the easiest practical way of obtaining the required length and breadth of a **rectangle** *equivalent to a given rectilineal figure is to obtain a* **triangle** *equivalent to it by the method of* Ex. 78, 79. *From the demonstration of* I. 42 *it easily follows* **that a rectangle with one side equal to half one of those of the triangle, and the other equal to the perpendicular let fall on it from the opposite angle, is equivalent to the triangle,** *and therefore to the given rectilineal figure.*

3. It is easy to see that the Proposition affords a simple solution to the problem (given as Ex. 93):—*To construct a square whose area shall be twice, three times, four times . . . a given square. We have only to* **make BE equal to the side of the given square, to produce BE to F, making EF equal to twice, three times, four times . . . that side,** *and to proceed as in the Proposition.*

Ex. 217.—To make a square equal to a given △ by the simplest construction possible.

Ex. 218.—To make a square equal to a given regular hexagon or octagon (see p. 85) by the simplest construction possible.

Ex. 218 (a).—To a given st. line (BE) to apply a rectangle (BEDC) equal to a given square.

Draw EH *perpr. to* BE *equal to a side of the given square.*

Let the perpr. bisector of BH *meet* BE *in* G, *and construct the figure of* II. 14. *Contrast the solution on pp.* 115, 116.

ON ABBREVIATIONS.

If a student about to enter for an examination in Geometry has any doubt as to the extent to which the use of symbols in Geometrical reasoning is allowed, application should be made to the examining authorities.

He may, we think, take it as a general rule that rather more freedom would be allowed in the solution of an exercise than in the reproduction of bookwork.

He should certainly, however, be acquainted with a few common abbreviations.

(1) The symbols + and − are used in the usual Arithmetical sense; $a \backsim b$ is used for 'the difference between a and b.'

(2) AB^2 is used as equivalent to 'sq. on AB.'

AB, BC	,,	,,	'rect. AB, BC.'
2 AB	,,	,,	'twice AB.'
n AB	,,	,,	'n times AB.'

We subjoin a few examples of the use of symbols.

Particular enunciations of II. 4 and II. 7—

Let a st. line AB be divided at C; then

$$AB^2 = AC^2 + CB^2 + 2\,AC, CB, \qquad [II. 4.$$
$$AC^2 = AB^2 + BC^2 - 2\,AB, BC. \qquad [II. 7.$$

Particular enunciation of II. 5 and II. 6—

If C is the mid-pt. of the st. line AB, and D any point in AB, or AB produced,

$$CB^2 \backsim CD^2 = AD, DB.$$

Ex. 219.—If the straight line PQ is divided in R, so that PQ, QR = PR^2, and PR is divided in S, so that PR, RS = PS^2, prove that PS = RQ. (Sandhurst: given thus, Dec. 1886.)

The student should practise himself in the use of the above symbols.

If he accustoms himself to quote a proved Geometrical theorem in justification of each step that he takes in demonstration, he will not be likely to impair the rigour of his investigations; while his knowledge of Algebra and its applications will be improved and strengthened. He will also find the results of Book II. more easy to remember when expressed symbolically.

Ex. 220.—Enunciate symbolically the remaining propositions of Book II.

NOTE.—In the Cambridge Locals :—

The only abbreviation admitted for 'the square on AB' is 'sq. on AB,' and for 'the rectangle contained by AB and CD,' 'rect. AB, CD.' All generally understood abbreviations or symbols for **words** may be used, but not symbols of **operations**, such as $-$, $+$, $\times$.

In the London University Matric. :—

The only abbreviations which can be permitted are sq. for 'square,' rect. for 'rectangle,' ‖gram for 'parallelogram,' $\angle$ for 'angle,' and the symbols $\therefore$, $\because$, $=$, and $+$ in their usual senses.

MISCELLANEOUS EXERCISES.—III.

(Book II.)

Ex. 221.—If O, A, B, C are four points in a straight line, in the order O, B, C, A, or O, A, C, B,

$$OA, BC+OB, CA=OC, AB.$$

Show that the Algebraical identity,

$$a(b-c)+b(c-a)+c(a-b)=0,$$

illustrates the truth of this theorem.

Ex. 222.—Find the locus of a point whose projection on a given fixed line is a given point.

Ex. 223.—To produce a given straight line so that the rectangle contained by the whole line thus produced, and the part produced, shall be equal to the square on half the given straight line.

Ex. 224.—The square on the perpendicular to the hypotenuse of a right-angled triangle from the right angle is equal to the rectangle contained by the segments into which it divides the hypotenuse.

Ex. 225.—A point O is taken in AB, such that $AO=3\ OB$. If P be any other point whatever, show that

$$AP^2+3\ BP^2=AO^2+3\ BO^2+4\ OP^2.$$

Apply II. 12 and 13 *to* $\triangle$*s* AOP, BOP, *and remember that rectangle contained by* AO *and projn. of* OP *will be three times that contained by* BO *and the same projn. by* II. 1.

Hence show that if $AP^2=4\ OP^2$, A and O being fixed points, the locus of P is a circle whose centre is in AO produced.

Ex. 226.—A point O is taken in AB such that $AO=\overline{n-1}\ OB$. If P be any other point whatever, show that

$$AP^2+\overline{n-1}\ BP^2=AO^2+\overline{n-1}\ BO^2+nOP^2.$$

Hence show that if $AP^2=nOP^2$ the locus of P is a circle whose centre is in AO produced, n being any integer greater than unity.

Ex. 227.—The rectangle contained by two given straight lines is equal to a given square. Show that if either the sum or the difference of the two straight lines is given, the lengths of the straight lines can be found. (Rouché and de Comberousse, *Traité de Géométrie Élémentaire.*) *Use* II. 8.

Ex. 228.—ABCD is a square whose diagonals intersect in O; AC is produced to E, so that CE is equal to AB; prove that the square on AE is double the square on OE.

Ex. 229.—Divide a line internally or externally into two parts such that the difference between their squares shall be equal to a given square.

Ex. 230.—If AB is divided at C so that the square on AC is double the square on CB, the sum of AB and CB will be equal to the diameter of the square on AB.

Ex. 231.—ABCDE is a straight line so divided that AB = BC = CD = DE, and O is an external point; show that the difference of the squares on OA and OE is twice the difference of the squares on OB, OD.

Ex. 232.—Describe a right-angled triangle such that the rectangle contained by the hypotenuse and one of the sides containing the right angle is equal to the square on the other side.

Ex. 233.—The sides AB, BC, CA of a △ ABC are 6, 10, and 14 units of length respectively; AB, CB are produced to meet the perprs. let fall on them from the opposite angles in D and E. Find the lengths of AD and CE.

Ex. 233 (a).—If a st. line be divided into any no. of parts the sq. on the whole line is equal to the sqs. on each of the parts with twice the sum of the rectangles contained by each pair of the parts.

Ex. 233 (b).—A, B, C, D are any four pts. in a st. line ABCD. Show that the sqs. on AC, BD with twice rect. AB, CD = sqs. on AD, BC (Leslie's *Elements*).

Ex. 233 (c.)—To divide a given st. line AB into two parts AH, HB, such that the sq. on AH may be equal to the rectangle contained by HB and a given finite st. line X (T. Simpson's *Elements*).

This is a generalisation of II. 11, *and may be solved in much the same way. Draw* AC *perpr. to* AB *and equal to* X. *Complete the rectangle* ABDC, *and along* AB *take* AL *such that sq. on* AL *rect.* = *rect.* ABDC. *Bisect* AC *in* E. *Join* EL. *Produce* EA *to* F *so that* EF = EL, *and describe a sq.* AFGH *on* AF.

Ex. 233 (d).—In the figure of II. 11 bisect AH in L, and shew that LG = LB.

Ex. 233 (e).—To produce a given st. line FA to a pt. C such that rect. AF, FC = sq. on AC.

Describe a sq. AFGH *on* FA. *Bisect* AH *in* L, *and construct the fig. of II.* 11 *from the property given in the last exercise.*

Ex. 233 (f).—Investigate the solution of Ex. 233 (e) by means of an algebraical equation.

Ex. 233 (g).—Enunciate the propositions represented by the equations

$$(2a \pm b)^2 + b^2 = 2\{a^2 + (a \pm b)^2\}$$

MISCELLANEOUS EXERCISES.—IV.

(Books I. and II.)

Ex. 234.—Given straight lines containing a, b, c, d units of length, construct lines containing

$$\text{(i.) } \frac{bc}{a},$$
$$\text{(ii.) } \sqrt{ab},$$
$$\text{(iii.) } \sqrt[4]{abcd},$$

units of length respectively.

Ex. 235.—Two triangles which have parallel sides are equiangular.

Ex. 236.—On the sides AB, BC, CA of a △ABC squares ABHL, BCGF, CAKI are described, and perpendiculars CO, AD, BM let fall on these sides (produced if necessary) from the opposite angle are produced to cut the parallel sides of the squares in P, E, N respectively. Show that rects. AOPL, BDEF, CMNI are respectively equal to rects. AMNK, BOPH, CDEG.

Hence obtain demonstrations of II. 12 and II. 13.

Ex. 237.—O is the orthocentre of the △ABC; show that sqs. on AO, BC = sqs. on BO, CA = sqs. on CO, AB.

Ex. 238.—The distance of the circumcentre from any side = half dist. of the orthocentre from the angle opposite to that side.

Ex. 239.—Show that if the straight line divided in II. 11 is twice the unit of length the two parts of the line are $\sqrt{5}-1$ and $3-\sqrt{5}$ respectively.

Ex. 240.—If in the fig. of I. 47 GH, KE, DF be joined, the three triangles GAH, KCE, DBF are equivalent. Is this the case when the triangle is not right-angled?

Ex. 241.—'The perprs. from the vertices of a △ to the opposite sides are concurrent.' (See Ex. 108.) Use this theorem for obtaining through a given point P a straight line which would pass through the intersection of two given straight lines AB, CD if all three were produced far enough, the point of intersection being inaccessible. (Rouché et de Comberousse, *Traité de Géométrie Élémentaire*.)

Ex. 242.—The median of a triangle is greater than, equal to, or less than half the side it bisects, according as the angle from which it is drawn is acute, right, or obtuse, and conversely.

Ex. 243.—One of the two sides containing the right angle of a right-angled triangle is double the other. Show that one of the segments into which the perpendicular from the right angle to the hypotenuse divides the base is four times the other. (R. & de C., *Traité de Géométrie Élémentaire.*)

Ex. 244.—The shortest median of a triangle bisects the longest side. (R. et de C.)

Ex. 245.—A triangle ABC is turned about the point A into the position AB′C′; if AC bisect BB′ prove that AB′, produced if necessary, will bisect CC′.

Ex. 246.—A parallelogram is inscribed in a second parallelogram, *the sides of the first being parallel to the diameters of the second.* Show that the diameters of the two parallelograms pass through the same point.

Examine the effect of omitting the words italicised.

Ex. 247.—A, B, C are taken on a straight line OABC, such that AB = BC. Show that 2 OB = OA + OC.

Ex. 248.—A, B, C are points in st. line OABC. D, E, F are the mid-points of BC, CA, AB. Show that

$$OD + OE + OF = OA + OB + OC.$$

Ex. 249.—A, B, C are points on a straight line OABC; D, E, F are the mid-pts. of AB, BC, and DE respectively. Show that

$$4\,OF = OA + 2\,OB + OC.$$

Ex. 250.—A, B, C, D are points in a straight line OABCD; E, F, G, H, K, L are the mid-pts. of AB, BC, CD, EF, FG, HK respectively. Show that

$$8\,OL = OA + 3\,OB + 3\,OC + OD.$$

Ex. 251.—**If three or more parallel straight lines intercept equal segments on one straight line that cuts them, they do so on all straight lines that cut them** (Syllabus). Hence show that a straight line may be divided into any given number of equal parts.

Ex. 252.—ABC are three points in a straight line such that AB = BC. Perprs. AD, BE, CF are drawn to any other straight line. If A, B, C are all on the same side of this line show that 2 BE = AD + CF.

If, instead of AB = BC, we had 2 AB = BC, show that, under similar conditions,

$$3\,BE = 2\,AD + CF.$$

Enunciate and prove a corresponding theorem with 3 AB = BC.

Ex. 253.—A, B, C are any three points on the same side of a straight line XY. AB is divided in D, so that AD = ½ AB. DC is divided in E, so that DE = ⅓ DC. Show that three times the distance of E from XY = the sum of the distance of A, B, C from the same line.

Ex. 254.—A, B, C, D are any four points on the same side of a st. line XY. AB is divided in E, so that AE $= \frac{1}{2}$ AB; EC is divided in F, so that EF $= \frac{1}{3}$ EC; FD is divided in G, so that FG $= \frac{1}{4}$ FD. Show that four times the distance of G from XY = the sum of the distances of A, B, C, D from XY. Also enunciate and prove a similar theorem with respect to five given points on the same side of a given straight line.

The final point of section is called the **centroid or centre of mean position** of the given points.

Ex. 255.—Show that the centroid of three, four, or five given points is independent of the order in which we take them. For a generalisation of the theorems given in the last four exercises, and some remarkable properties of the **centroid**, see Casey's *Sequel to Euclid*.

Ex. 256.—Find the locus of a point the sum of the squares of whose distances from three given points is constant.

Def.—**If points P and Q be taken on the sides AB, AC respectively of △ ABC, making ∠ APQ = ∠ C, and ∴ ∠ AQP = ∠ B, the st. line PQ is said to be 'Anti-parallel' to BC with respect to A.**

Ex. 257. **Show that the locus of the mid-points of anti-parallels to BC with respect to A is a straight line through A.** (See Ex. 109.) **This straight line is called a 'Symmedian Line,' or a 'Symmedian' of the △ ABC.**

There will clearly be three such lines, one through each angular point bisecting the anti-parallels to the opposite side.

Ex. 258.—**The three symmedian lines of a triangle are concurrent.**

Let the symmedians of the △ ABC through A and B intersect in K; draw the anti-parallels PQ (to BC with respect to), RS (to CA with respect to B), TV (to AB with respect to C), through K.

Then ∠ KQT = ∠ B (∵ PQ is anti-parallel to BC).
= ∠ KTQ (∵ TV „ AB).
∴ KT = KQ.
Similarly KS = KP,
and KV = KR.
But KP = KQ (∵ K is on the symmedian through A), } Ex. 257.
and KR = KS (∵ K „ „ „ B), }
∴ KT = KV.
∴ K lies on the symmedian through C.

This point is called the 'Symmedian Point' of the triangle.

Cor.—**A circle can be described with centre K to pass through the six points P, Q, R, S, T, V.**

This circle is called the 'Cosine Circle' of the triangle.

Ex. 259.—To inscribe three rectangles in a triangle whose diagonals shall all intersect in the same point. The previous exercise affords a solution of this problem, the point of intersection being the symmedian point of the triangle.

Ex. 260.—Each of the straight lines which join the mid-point of the side of a triangle to the mid-point of the perpr. let fall on it from the opposite angle passes through the symmedian point of the triangle.

Ex. 261.— AD, BE, CF are perprs. from A, B, C to the sides BC. CA, AB of a △ ABC. Perprs. are also drawn to BC, CA, AB through points d, e, f respectively, such that the mid-pts. of the sides are also the mid-pts. respectively of Dd, Ee, Ff; show that these perprs. are concurrent. (R. Tucker in the *Educational Times.*)

Prove (i.) *by* Ex. 154, (ii.) *by* Ex. 108, *using a diagram like the one given; the pt. of concurrence will be the orthocentre of the dotted* △.

Ex. 262.—Perprs. PD, PE, PF are drawn from a point P to the sides BC, CA, AB of a △ ABC, and d, e, f are taken as in the last Exercise. Show that the perpendiculars to BC, CA, AB, through d, e, f respectively are concurrent.

Ex. 263.—With the construction of Ex. 261 show that Ad, Be, Cf are concurrent. (R. Tucker in the *Educational Times.*)

Use Ex. 108, *and a diagram like the one given.*

The pt. of concurrence is the symmedian point of the dotted △. *See* Ex. 261.

Ex. 264.—Given a point within a triangle, inscribe three parallelograms in the triangle having their diagonals all passing through the point.

Use Ex. 122.

Ex. 264 (a).—If in Ex. 254 A, B, C, D are the corners of a ||gm. show that the diagls. AC, BD of the ||gm. cross at G.

Ex. 264 (b).—ABCD being any quadl. and P, Q, R, S the mid-pts. of DA, AB, BC, CD show that PR, QS bisect each other at the pt. G found as in Ex. 254.

Ex. 264 (c).—**The square on any finite straight line is equal to the sum of the squares of its projections on any two straight lines at right angles to each other.**

Ex. 264 (d).—In the fig. of I. 47, shew that sq. on FC is greater than sqs. on AB, BC by four times △ ABC.

THE HARPUR EUCLID.

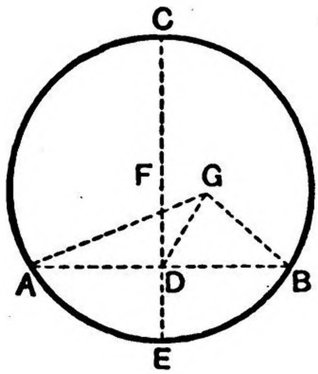
BOOK III.

PROPOSITION 1.

To find the centre of a given circle.

Let ABC be the given ⊙; it is required to find its centre.

Take any two pts. A and B on the ⊙; join AB and bisect it at D; through D draw CE ⊥r to AB, meeting the ⊙ at C and E; bisect CE at F; F shall be the centre.



Take any pt. G within the ⊙, but not in CE, and join GA, GD GB.

In the two △s ADG, BDG

AD=BD,

GD is common,

and ∠ ADG is unequal to ∠ BDG;

∴ GA is unequal to GB, [I. 24.

∴ G is not the centre.

In the same way it may be shown that any other pt. within the ⊙ outside CE is not the centre;

∴ F, the mid-pt. of CE, must be the centre.

Corollary.—From this it is clear that

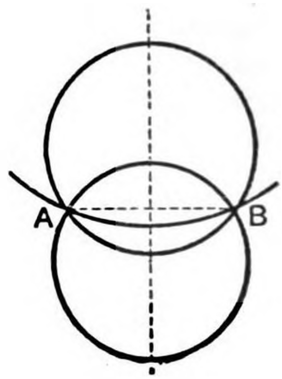
If in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

In other words—

If a circle pass through two given points, its centre lies on the perpendicular bisector of their join.

Hence

If any number of circles pass through two given points their centres all lie on the perpendicular bisector of the join of the two given points.



NOTES.

When a straight line is spoken of as 'drawn within a circle,' it is implied that its extremities are on the circumference.

Any straight line, such as AB, drawn within a circle, is called a **chord** of the circle. Hence the Corollary might be enunciated thus :—

The perpendicular bisector of any chord of a circle passes through the centre of the circle.

Ex. 265.—A circle cannot have two centres.

Ex. 266.—Circles whose centres are A and B intersect in C. Through C draw a line PCQ parallel to AB, terminated by the circles.

Show that PQ = 2 AB.

The perpendicular bisectors of CP, CQ *pass through* A *and* B *respectively.*

Ex. 267.—ABCD is a quadrilateral inscribed in a circle. Show that the perpendicular bisectors of its sides and its diagonals are concurrent.

Extend the theorem to other rectilineal figures.

Ex. 268.—A, B, C, D are four points. Show that if the perpr. bisectors

of AB, AC, AD are concurrent, so also are the perpendicular bisectors of BC, BD, CD.

Extend the theorem to a system of five points.

It will be noticed that the word 'circle' is used in two different senses. Sometimes it denotes *the figure bounded by the circumference*, as when we speak of *the segment of a circle* (see III. 21); sometimes *the circumference itself*, as when we speak of *a circle passing through two given points*.

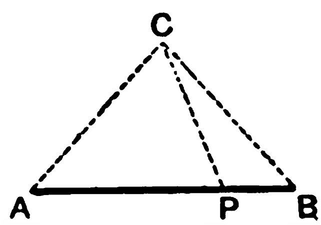
According to Euclid's definition it should be used in the first sense only, but we are not aware of any writer whose use of the word is thus strictly limited. Euclid himself, for instance, speaks (I. 1) of a point where *two circles cut each other*, when he means a point *where their circumferences cut*. No confusion, however, can well happen from this double use of the word, and it is certainly convenient in many cases to avail ourselves of it. There are several other instances in Mathematics of this double use of a word in cases where no mistake is likely to follow from it.

The symbol for circumference is '⊙ce.'

PROPOSITION 2.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let A and B be two pts. on the ⚪ce of a ⊙.
Their join, AB, shall be within the ⊙.



Let C be the centre of the ⊙, and P any point in AB. Join CA, CP, CB.

Rad. CA = rad. CB,
∴ ∠ CAB = ∠ CBA.
But ext. ∠ CPA > int. and opp. ∠ CBA,
∴ ∠ CPA > ∠ CAB,
∴ CA > CP;
i.e. CP < radius of the ⊙.
∴ P lies within the ⊙.

Similarly it may be shown that any other pt. in AB, except its extremities, lies within the ⊙,
∴ AB lies within the ⊙.

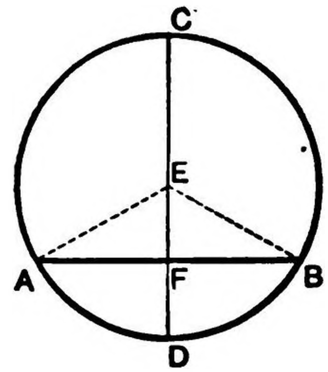
Ex. 269.—The extremities of the base of an isosceles triangle are farther from the vertex than any other point in the base.

PROPOSITION 3.

(1) If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre it shall cut it at right angles.

(2) If a straight line drawn through the centre of a circle cut a straight line in it which does not pass through the centre at right angles it shall bisect it.

(1) Let ABC be a ⊙, and let a st. line CD, passing through the centre E, bisect a st. line AB which does not pass through the centre, at F, then CD is ⊥r to AB.



Join EA, EB.

In the △s AFE, BFE

AF=BF,

EF is common,

and rad. EA=rad. EB,

∴ ∠ AFE= ∠ BFE,

i.e. CD is ⊥r to AB.

(2) Next let CD, passing through the centre E, cut AB at rt. ∠s at F, then AF=FB.

With the same construction,

rad. EA =rad. EB,

∴ ∠ EAB= ∠ EBA.

In the two $\triangle$s EAF, EBF,

$\angle$ EAF $=$ $\angle$ EBF,

rt. $\angle$ EFA $=$ rt. $\angle$ EFB,

and side EF, opposite equal $\angle$ s in each, is common,

$\therefore$ AF $=$ FB.

NOTES.

1. Since the triangle EAB must be *isosceles*, we have only to demonstrate that

(i.) The straight line joining the vertex of an isosceles triangle to the mid-point of its base is perpendicular to the base.

(ii.) The perpendicular from the vertex of an isosceles triangle to the base bisects the base.

2. The words 'which does not pass through the centre' might obviously be omitted from the second part of the enunciation without affecting its truth. Hence

Every diameter of a circle is an axis of symmetry (see p. 23).

Ex. 270.—**The mid-points of a set of parallel chords of a circle lie in a straight line.**

Ex. 271.—If a trapezoid be inscribed in a circle it must be symmetrical about one of its medians.

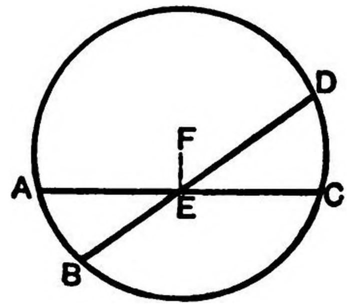
In other words it must be an axe. (See pp. 103 and 106.)

Ex. 272.—An axe can always have a circle described about it.

PROPOSITION 4.

If in a circle two straight lines cut each other which do not both pass through the centre they do not bisect each other.

Let ABCD be a ⊙, and let AC, BD, two st. lines in it, cut each other in a pt. E which is not the centre; then AC, BD do not bisect each other.



For, *if possible*, let AE=EC and BE=ED.

Find the centre F of the ⊙.

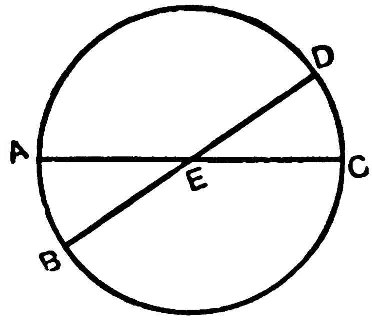
Neither AC nor BD can pass through F, or it would be bisected at F instead of at E. Join FE.

∵ AE=EC,
∴ FE is ⊥r to AC; [III. 3.
∵ BE=ED,
∴ FE is ⊥r to BD; [III. 3.
∴ rt. ∠ FEA=rt. ∠ FEB,
which is absurd.

Alternative Proof.—Let the st. lines AC, BD in the ⊙ ABCD bisect each other at E; then E shall be the centre.

The st. lines through E ⊥r to AC, BD must each pass through the centre;

∴ the centre must be at E, the only pt. common to these ⊥rs.



∴ if two st. lines in a ⊙ bisect each other, they both pass through the centre.

∴ if two st. lines in a ⊙ do not both pass through the centre, they do not bisect each other.

NOTE.

The student will perhaps have already noticed in Book I. pairs of Propositions connected with each other in the same way as the one enunciated as III. 4, and the one directly demonstrated in the Alternative Proof. The general relationship between such a pair of Propositions may be thus stated :—

The two Propositions,

If A is B then C is D,
If C is not D then A is not B,

are each a necessary consequence of the other, *i.e.* if *either* is true the *other* must also be true.

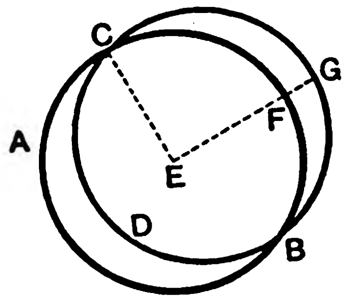
Each of the above theorems is called the **contrapositive** of the other. (Syllabus.)

In many cases we can show the truth of a Proposition which Euclid proved indirectly by demonstrating the contrapositive theorem *directly*.

PROPOSITION 5.

If two circles cut one another they shall not have the same centre.

Let the two ⊙s ABC, DCG cut each other at C, they shall not have the same centre.



Find the centre E of the ⊙ ABC; join EC, and draw any st. line EFG, cutting the ⊙s in F and G.

∴ Rad. EF=rad. EC,

∴ EG is not equal to EC,

∴ E is not the centre of the ⊙ DCG.

Ex. 273.—Write down the *contrapositive* of III. 5.

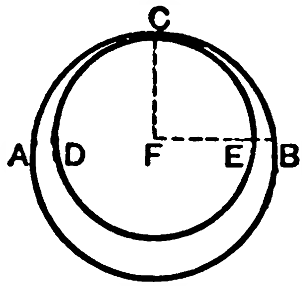
Def.—Circles are said to touch one another which meet, but do not cut another.

If each of two circles which touch each other is outside the other, they are said to **touch each other externally**; if one is wholly within the other, they are said **to touch each other internally.**

PROPOSITION 6.

If two circles touch one another internally they shall not have the same centre.

Let the ⊙s ABC, CDE touch one another internally at C, they shall not have the same centre.



Find F, the centre of the inner ⊙ CDE; join FC, draw FE any other radius of CDE, and produce it to cut ABC in B.

Rad. FE = rad. FC,

∴ FB is not equal to FC,

∴ F is not the centre of ABC.

Ex. 274.—Write down the contrapositive of III. 6.

Def.—**If two circles have the same centre, they are said to be concentric.**

From III. 5 and III 6 we see that

If two circles have a common point, they cannot be concentric.

Hence, *contrapositively*,

If two circles are concentric, they cannot have a common point.

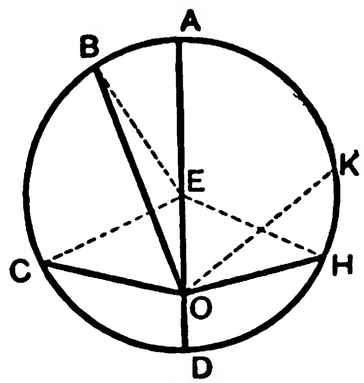
Ex. 275.—A chord AD of the larger of two concentric circles cuts the smaller in B and C. Show that AB = CD. Show also that AB and CD subtend equal angles at the centre.

PROPOSITION 7.

(1) If a point be taken on the diameter of a circle which is not the centre, of all straight lines that can be drawn from it to the circumference the greatest is that in which the centre is, and the remaining part of the diameter is the least, and of the others that which is nearer to the one through the centre is greater than one more remote.

(2) From the same point there can be drawn two straight lines, and only two, that are equal to each other, one on each side of the diameter.

Let O be a pt. on the diameter AD of a ⊙ ABC which is not the centre.



(1) Of all st. lines which can be drawn from O to the ○ce the greatest is OA, which passes through the centre E, and OD is the least; of two others OB, OC, that one, OB, which is nearer to OA shall be the greater.

Join EB, EC.

Rad. EA = rad. EB,

∴ OA = OE, EB (by adding OE).

But OE, EB > third side OB,

∴ OA > OB.

Again, rad. ED = rad. EB,

∴ ED < EO, OB,

∴ remr. OD < remr. OB.

Similarly, it may be shown that OA is greater and OD less than any other st. line drawn from O to the ○ce.

∴ OA is the greatest and OD the least of all st. lines from O to the ○ce.

Again, in △s OEB, OEC,

OE, EB = OE, EC,

and ∠ OEB > ∠ OEC,

∴ OB > OC.

(2) One st. line, and only one, can he drawn from O to the ○ce equal to any st. line OC already drawn.

Draw a radius EH, such that ∠ OEH = ∠ OEC. Join OH.

In △s OEH, OEC

OE, EH = OE, EC,

and ∠ OEH = ∠ OEC,

∴ OH = OC.

Let any other st. line OK be drawn from O to the ○ce on the same side of AD as OH.

Then OK either > or < OH by (1),

∴ OK either > or < OC;

∴ no other st. line can be drawn from O to the ○ce which = OC.

N.B.—By the *construction* ∠ OEH = ∠ OEC,

not ∠ EOH = ∠ EOC

as students often put by mistake.

Ex. 276.—**If two equal straight lines be drawn from any point which is not the centre to the circumference, they will be equally inclined to the diameter through the point.**

Ex. 277.—Show by III. 7 that a circle is symmetrical with respect to any diameter.

The student would perhaps find it as well to omit III. 8 on his first reading.

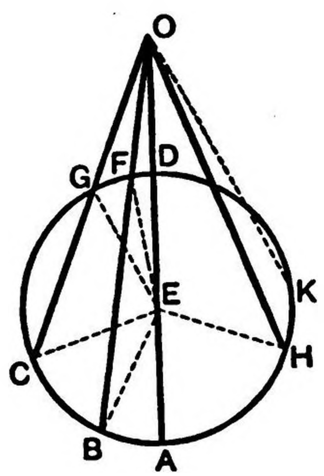
PROPOSITION 8.

(1) If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre; of those which fall on the concave circumference the greatest is the one through the centre, and of the rest that which is nearer to the one through the centre is always greater than one more remote.

(2) Of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote.

(3) From the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.

Let O be a pt. without the ⊙ ABC.



(1) Of all st. lines which can be drawn from O to the concave Oce the greatest is OA, which passes through the centre E, and of two others, OB, OC, that one OB which is nearer to OA shall be the greater.

Rad. EA = rad. EB,

∴ OA = OE, EB.

But OE, EB > 3rd side OB,
∴ OA > OB.

Similarly it may be shown that OA is greater than any other st. line drawn from O to the concave Oce,

∴ OA is the greatest of all such lines.

In $\triangle$s OEB, OEC $\begin{cases} \text{OE, EB = OE, EC,} \\ \text{and } \angle\text{OEB} > \angle\text{OEC,} \end{cases}$
∴ OB > OC.

(2) Of all st. lines which can be drawn from O to the convex part of the Oce, the least is OD, the part of OA without the ⊙, and of two others, OF, OG, that one OF which is nearer to OA shall be the less.

EO < EF, FO,
and rad. ED = rad. EF,
∴ OD < OF.

Similarly it may be shown that OD < any other st. line drawn from O to the convex Oce.

∴ OD is the least of all such lines.

In $\triangle$s OEF, OEG $\begin{cases} \text{OE, EF = OE, EG,} \\ \text{and } \angle\text{OEF} < \angle\text{OEG,} \end{cases}$
∴ OF < OG.

(3) One st. line, and only one, can be drawn from O to the Oce equal to any st. line OC already drawn.

Draw a radius EH such that $\angle$OEH = $\angle$OEC. Join OH.

In $\triangle$s OEH, OEC $\begin{cases} \text{OE, EH = OE, EC,} \\ \text{and } \angle\text{OEH} = \angle\text{OEC,} \end{cases}$
∴ OH = OC.

Let any other st. line OK be drawn from O to the Oce on the same side of AD as OH.

Then OK either > or < OH by (1) and (2),
∴ OK either > or < OC.

N.B.—By the construction $\angle OEH = \angle OEC$
not $\angle EOH = \angle EOC$.
Compare III. 7.

Ex. 278.—A, B are points without a circle. Find a point C on the circumference such that the sum of the squares on AC, CB is the least possible.

Let E *be the centre and* O *the mid-point of* AB ; *the pt. reqd. will lie on* EO *by* Ex. 203.

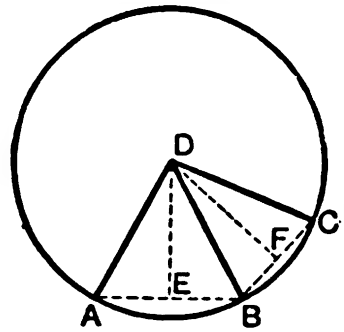
Find C also so that the sum of the squares shall be a maximum.

Ex. 279.—Find the greatest and least lines which can be drawn from any point on the circumference of a circle to any point on the circumference of another which does not meet it.

Any three sides of a quadrilateral are together greater than the fourth.

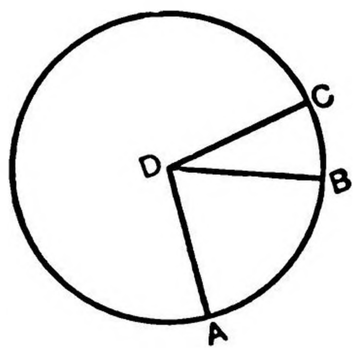
PROPOSITION 9.

If a point be taken within a circle from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.



Let there be a pt. D within the $\odot$ ABC such that three equal st. lines DA, DB, DC can be drawn from it to the Oce; then D shall be the centre.

Join AB, BC; bisect AB, BC in E, F, and join DE, DF.



In $\triangle$s DEA, DEB

DE, EA = DE, EB,

DA = DB,

$\therefore$ $\angle$ DEA = $\angle$ DEB;

i.e. DE is $\perp$r to AB;

$\therefore$ the centre lies in ED. [III. 1. COR.

Similarly it can be shown that the centre lies in FD,

$\therefore$ it must be at D, the only pt. common to ED, FD.

Alternative Proof.—[Draw a diameter through D.]

Then if D were not the centre there could not be more than two equal st. lines drawn from D to the Oce, [III. 7.

$\therefore$ D is the centre.

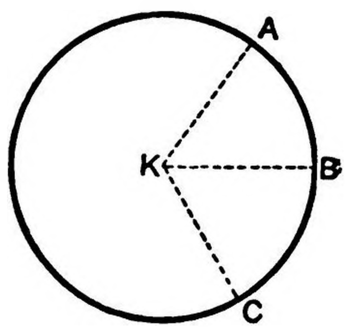
NOTE.

The first proof is one of Euclid's own, but is not given by Simson.

PROPOSITION 10.

One circumference of a circle cannot cut another at more than two points.

If it be possible let one ○ce cut another at the three pts. ABC.



Find the centre K of one of the two ⊙s and join KA, KB, KC.

Then rad. KA=rad. KB=rad. KC,

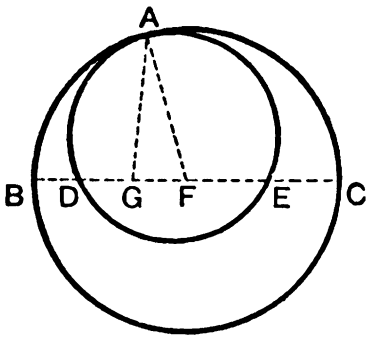
∴ K is the centre of the other ⊙, [III. 9.

which is impossible. [III. 5.

PROPOSITION 11.

If two circles touch each other internally the straight line which joins their centres, being produced, shall pass through the point of contact.

Let the two ⊙s ABC, ADE touch each other internally at the pt. A; the st. line through their centres shall pass through A.



Find F, the centre of the outer ⊙ ABC.
Join F to any pt. G within ⊙ ADE which is not on FA.
Produce FG to cut the ○ces in D, B.
Join GA.
Then GA>GB, [III. 7.
∴ GA>GD;
∴ G is not the centre of ⊙ ADE.
Similarly it can be shown that, if FG cut the ⊙s at any other pt. than A, G is not the centre of ⊙ ADE.
∴ the st. line joining F to centre of ⊙ ADE passes through A when produced.

NOTE.

It follows from the above demonstration that there cannot be a second point of internal contact. For if there were a second one, K, the centre of ADE, would also lie on the line FK, which is impossible, as the two circles have not the same centre.

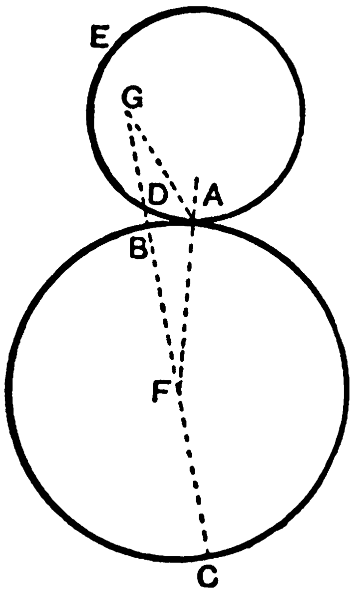
Ex. 280.—Two circles touch each other internally. Show that if a st. line be drawn perpr. to the diameter through the point of contact the two parts of it lying between the two circumferences are equal.

Ex. 281.—Enunciate and prove a converse of the preceding exercise.

PROPOSITION 12.

If two circles touch one another externally the straight line which joins their centres shall pass through the point of contact.

Let the two ⊙s ABC, ADE touch each other externally at the pt. A; the join of their centres shall pass through A.



Find the centre F of ⊙ ABC.
Join F to any pt. G within ⊙ ADE which is not on FA produced.
Let FG cut the Oces in B, D. Join GA.
Then GA>GB; [III. 8.
∴ GA>GD.
∴ G is not the centre of ⊙ ADE.
Similarly it can be shown that if FG cut the Oces at any other pt. than A, G is not the centre,
∴ the st. line joining F to the centre of ⊙ ADE passes through A.

NOTE.

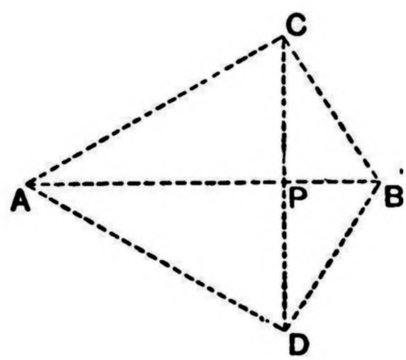
It follows from the above demonstration that there cannot be a second point of external contact. For if there were a second one, K, the centre of ⊙ ADE, would also lie on FK produced, which is impossible.

Observe the close resemblance between the above demonstration and that of III. 11.

Alternative Proof.—Let two ⊙s, whose centres are A and B, **meet** at a pt. C which is not on AB; they shall not touch each other externally.

Draw CP ⊥r to AB and produce it to D, so that PD = PC.

Join AC, AD, BC, BD.



In △s APC, APD

AP, PC = AP, PD,

and ∠APC = ∠APD;

∴ AC = AD;

∴ D lies on the ⊙ whose centre is A.

Similarly D can be shown to lie on the circle whose centre is B,

∴ CD lies in both ⊙s;

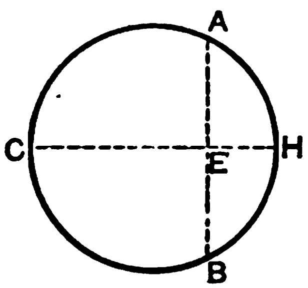
∴ they do not touch externally;

∴ if two ⊙s do touch externally they cannot meet at a pt not on the line joining the centres.

PROPOSITION 13.

One circle cannot touch another in more points than one, either internally or externally.

If possible let one ⊙ touch another ⊙ ABC in the pts. A, B, either internally or externally.



Join AB. Bisect AB in E, and draw CEH ⊥r to AB.

Then AB lies in each ⊙, [III. 2.

∴ the centres of both ⊙s lie on CEH or CEH produced, [III. 1.

i.e. the line joining the centres of two ⊙s which touch does not pass through either of the pts. of contact, A, B, which is absurd.

Alternative Proof.—∵ the line joining the centres must pass through each pt. of contact,

∴ AB must be that line,

∴ AB has two mid-pts. (viz. the centres of the two ⊙s); which is absurd.

NOTES.

It will be convenient at this stage to summarise the results obtained as regards the different relative positions possible to two circles. It follows from III. 11 that

(1) **If two circles touch internally the difference of their radii is equal to the join of their centres,**

and from III. 12 that

(2) **If two circles touch externally the sum of the radii is equal to the join of their centres,**

and the student should have little difficulty in showing that

(3) **If one circle lies within another without meeting it the difference, and therefore also the sum of their radii, is greater than the join of the centres.**

(4) **If each of two circles lies outside the other without meeting it the sum, and therefore also the difference of their radii, is less than the join of their centres.**

(5) If two circles cut each other the difference of their radii is less but the sum greater than the join of their centres.

We may exhibit these results conveniently by means of the following table, which we take with slight alteration from Henrici's *Congruent Figures.*

Let R, r be the radii of the two circles, and suppose $R > r$; and let D be the distance between the centres.

HYPOTHESIS.	CONCLUSION.
(*a*) **If one ⊙ lies inside the other not meeting it**	$R - r > D.$
(*b*) **If the two ⊙s touch internally**	$R - r = D.$
(*c*) **If the two ⊙s cut**	$R - r < D$; $R + r > D.$
(*d*) **If the two ⊙s touch externally**	$R + r = D.$
(*e*) **If each ⊙ lies outside the other not meeting it**	$R + r < D.$

The converse of each of the above theorems is true.

As an example let us take the converse of (*e*).

If the sum of the radii of two unequal circles is less than the join of their centres each must lie outside the other without meeting it.

For if not they must either touch or cut each other, or one must lie entirely within the other, and in each of these cases the sum of their radii is not less than the join of their centres, by (1), (2), (3), and (4).

The above is an example of the application of the *Rule of Conversion.*

If of the hypotheses of a group of demonstrated theorems it can be said that one must be true, and of the conclusions that no two can be true at the same time, then the converse of every theorem of the group will necessarily be true. (Syllabus.)

Euclid's treatment of the contact and intersection of circles is rather unsystematic. The demonstration of III. 10 which we have followed appears defective, for it is not shown that K lies *within* the second circle. It may be noticed, however, that K cannot be without that circle by III. 8., and that the demonstration of III. 7, *so far as regards the possibility of drawing three equal straight lines from a point to the circumference,* really applies to a point *on* the circumference as well as to a point within the circle. All these difficulties might be avoided thus :—

Let a circle pass through the three points A, B, C: then its centre must lie at the intersection K of the perpendicular bisectors of AB, BC; there can therefore be only one such circle; for two circles which cut cannot have the same centre. The alternative demonstration of III. 9 shows clearly that the restriction *within the circle* in the enunciation is unnecessary.

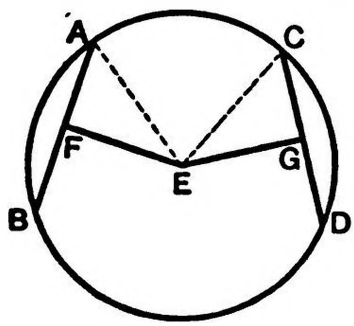
DEF.—**Straight lines are said to be equally distant from the centre of a circle when the perpendiculars drawn to them from the centre are equal.**

PROPOSITION 14. THEOREM.

(1) Equal straight lines in a circle are equally distant from the centre.

(2) Straight lines in a circle which are equally distant from the centre are equal.

Let AB, CD be chords of a ⊙ whose centre is E, and let EF, EG be drawn at rt. ∠s to AB, CD and ∴ bisecting them in F, G. Join AE, EC. [III. 3.



(1) If AB=CD,

then EF=EG,

∵ AB=CD,

∴ AF=CG. [AB, CD are bisected.

Now sqs. on EF, FA=sq. on EA, [I. 47.

=sq. on EC, [∵ EC=EA.

=sqs. on EG, GC. [I. 47.

But sq. on FA=sq. on CG. [∵ AF=CG.

∴ sq. on EF=sq. on EG,

∴ EF=EG.

(2) If EF = EG,

then AB = CD,

For sqs. on EF, FA = sqs. on EG, GC. [As in (1).

But sq. on EF = sq. on EG, [∵ EF = EG.

∴ sq. on FA = sq. on GC,

∴ FA = GC,

∴ AB = CD.

Ex. 282.—**The mid-points of a set of equal chords of a circle lie on a concentric circle.**

Ex. 283.—**Prove III. 14 by showing the congruence of the triangles EAB, ECD.**

Note that superposition could be effected by rotating the triangle ECD *until* EC *fell along* EB.

Ex. 284.—In the figure of III. 14 join AC, BD and show that AC is parallel to BD.

Ex. 285.—Those chords of a circle which are bisected by a circle concentric with it are equal.

Ex. 286.—On two equal chords AB, CD of a ⊙ whose centre is E, points H and K are taken such that AH = CK. Show that EH = EK.

Ex. 287.—A ⊙ intercepts parts PQ, RS of two equal chords AB, CD of a concentric ⊙. Show that PQ = RS.

Ex. 288.—Enunciate and prove a converse of Ex. 287.

Ex. 289.—Through a given point within a circle to draw a chord equal to a given chord.

(*A hint may be obtained from* Ex. 287.)

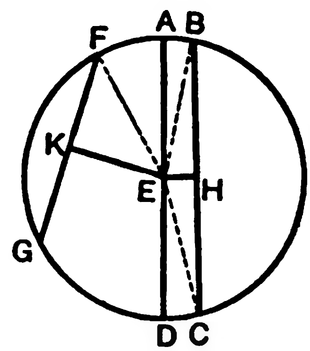
Ex. 289 (*a*).—Equal chords of a ⊙ which cross divide each other into equal segments, the greater to the greater and the less to the less.

DEF.—When the perpendiculars from the centre on two straight lines in a circle are not equal, that on which the greater perpendicular falls is said to be further from the centre than the other.

PROPOSITION 15. THEOREM.

(1) The diameter is the greatest straight line in a circle.
(2) Of all others, that which is nearer to the centre is greater than one more remote.
(3) The greater is nearer to the centre than the less.

Let ABCD be a ⊙ of which AD is a diameter and E the centre; and of the two chords BC, FG let BC be the nearer to E.



(1) AD > than any chord BC which is not a diameter.
(2) BC > FG.

Draw EH, EK at rt. ∠s to BC, FG and ∴ bisecting them at H, K.

Join EB, EC, EF.

∵ Radii AE, ED = radii BE, EC,
∴ AD = BE, EC.

But BE, EC > BC,
∴ AD > BC.

Similarly it might be shown that AD is greater than any other chord which is not a diameter.

Again, sqs. on EH, HB = sq. on EB, [I. 47.
= sq. on EF, [∵ EB = EF.
= sqs. on EK, KF. [I. 47.
But sq. on EH < sq. on EK, [∵ EH < EK.
∴ sq. on HB > sq. on KF,
∴ HB > KF,
∴ BC > FG.

(3) Next let chord BC > chord FG,
then with the same construction EH < EK.
∵ BC > FG,
∴ HB > KF,
∴ sq. on HB > sq. on KF.
But as in (2) sqs. on EH, HB = sqs. on EK, KF,
∴ sq. on EH < sq. on EK,
∴ EH < EK.

Ex. 290.—**Prove III. 15 by supposing △ EBC to rotate about E until EH fell along EK, and using I. 24.**

Ex. 291.—In the figure of III. 15 join EG and show that ∠ BEC > ∠ FEG.

Ex. 292.—**One chord of a circle is greater than, equal to, or less than another, according as the angle subtended at the centre by the first is greater than, equal to, or less than the angle subtended at the centre by the second.**

Ex. 293.—Through the mid-points of two equal chords of a ⊙ a third chord is drawn; show that the portions of this chord intercepted between the equal chords and the Oce are equal.

Ex. 294.—**Through a given point within a circle draw the shortest possible chord.**

Def.—**A straight line is said to touch a circle when it meets the circle and being produced does not cut it.**

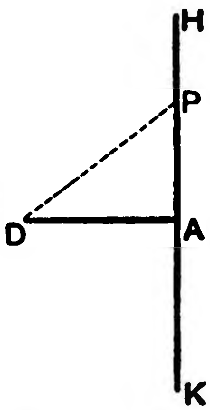
Such a straight line is called **a tangent to the circle.**

A straight line which cuts a circle is called a **secant.**

PROPOSITION 16. Theorem.

The straight line drawn at right angles to the diameter of a circle from the extremity of it falls without the circle, and no straight line can be drawn from the extremity between that straight line and the circumference so as not to cut the circle.

(1) Let D be the centre of a ⊙, A a pt. on its ○ce, and through A let the st. line HAK be drawn ⊥r to DA, then HAK shall fall without the ⊙.



Take any pt. P on HK and join DP.

∵ ∠ DAP is a rt. ∠,

∴ ∠ DPA < a rt. ∠,

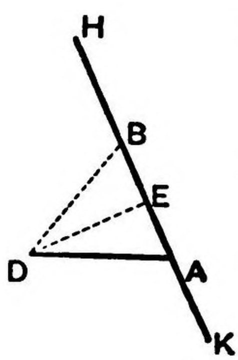
∴ ∠ DAP > ∠ DPA,

∴ DP > DA,

∴ P lies without the ⊙.

Similarly it might be shown that any other pt. except A on HK lies without the ⊙.

(2) Next let the st. line HAK drawn through the end A of a radius DA of a ⊙ whose centre is D be not ⊥r to DA.



Draw DE ⊥r to HK, and along HK on the side remote from A cut off EB equal to EA, and join DB.

In △s DEB, DEA

DE, EB = DE, EA,

and rt. ∠ DEB = rt. ∠ DEA.

∴ DB = DA.

∴ B lies on the ○ce,

∴ AB falls within the ⊙,

and ∴ HK cuts the ⊙.

COROLLARY 1.—**The straight line drawn at right angles to the diameter of circle, through the end of it, touches the circle.**

COROLLARY 2.—**A straight line cannot touch a circle in more than one point.**

(*For if a st. line met the* ○*ce in two pts., that part of the line between those two pts. would lie within the circle.*)

COROLLARY 3.—**There can be but one straight line which touches the circle at the same point.**

NOTES.

If all lines in a set are tangents to a curve it is said that the lines **envelop** that curve, and the curve is said to be the **envelope** of the set.

Thus we have shown in III. 16 (1) that

The straight lines equidistant from a fixed point envelop a circle which has the fixed point for its centre, and the constant distance for its radius. Henrici, *Congruent Figures*, pp. 169, 170.

Ex. 295.—**All equal chords of a circle are touched by the concentric circle which passes through their mid-points.**

Enunciate this as a proposition on 'envelopes.'

Ex. 296.—**The tangents to a circle through the ends of one of its diameters are parallel.**

Ex. 297.—Enunciate and prove the converse of the last Ex.

Ex. 298.—**The locus of the centre of a circle which touches a given straight line at a given point is the perpendicular to it through the given point.**

Ex. 299.—To draw a circle which shall touch a given straight line at a given point, and pass through another given point.

Ex. 300.—Any number of circles can be drawn to touch two given parallel straight lines.

Ex. 301.—The circles whose centres are on a given straight line, and whose radii are all of the same given length, envelop a pair of lines parallel to the given straight line.

Ex. 302.—Any number of circles can be drawn to touch two given concentric circles.

Ex. 303.—The circles whose centres are on a given ⊙, and whose radii are all of the same given length, envelop *in general* two circles concentric with the given one.

When does one of these two circles become a point?

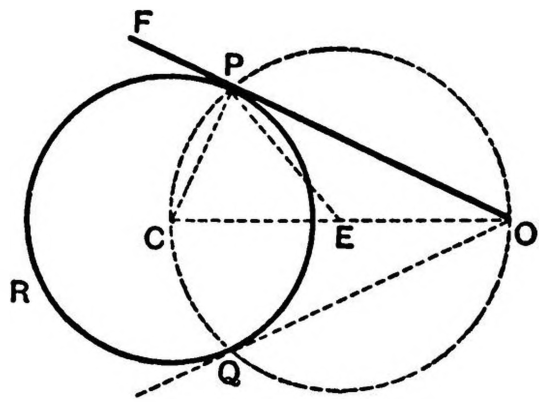
PROPOSITION 17. PROBLEM.

To draw a straight line from a given point either without or on the circumference which shall touch a given circle.

(1) Let O be a pt. without the given ⊙ PQR; it is required to draw a tangent to PQR from O.

Find the centre C of ⊙ PQR. Join OC, and bisect OC in E; with centre E and distance EC or EO describe a ⊙ cutting PQR in P. Join OP. OP shall be a tangent to PQR.

Produce OP to F Join EP, PC.



∵ EP = EO
∴ ∠ EPO = ∠ EOP.
Similarly ∠ EPC = ∠ ECP.
∴ whole ∠ CPO = ∠s EOP, ECP.
= ext. ∠ CPF.
∴ each is a rt. ∠
∴ OP touches the ⊙ at P. [III. 16.

(2) If the given pt. be on the ○ce as P, then the tangent required is the line PO drawn at rt. ∠s to the radius CP.

NOTES.

1. If the two circles cut again at Q, OQ would also be a tangent to ⊙ PQR. We have therefore demonstrated that

From an external point O two tangents can be drawn to a given circle.

2. **The two tangents OP, OQ which can be drawn from an external point O to a given circle are equal.**

In the figure of III. 17. Join CQ, PQ.

$$\because \quad CP = CQ,$$
$$\therefore \angle CPQ = \angle CQP.$$
$$\text{But rt. } \angle CPO = \text{rt. } \angle CQO,$$
$$\therefore \text{remg. } \angle OPQ = \text{remg. } \angle OQP,$$
$$\therefore OP = OQ.$$

3. The join CO of an external pt. O, and the centre C, bisects the angle between the two tangents OP, OQ which can be drawn from O, and also the angle between the radii CP, CQ.

For the three sides CO, OP, PC of △ COP = the three sides CO, OQ, QC of △ COQ.

Euclid does not demonstrate the equality of the two tangents to a circle from an external point until he reaches IV. 12. This is a serious omission. Students will often require the property when attempting exercises, and should demonstrate it if they are not supposed to have read IV. 12.

We give alternate demonstration which does not require joining PQ.

$$\text{Sqs. on } CP, PO = \text{sq. on } CO, \qquad [\text{I. } 47.$$
$$= \text{sqs. on } CQ, QO.$$
$$\text{But sq. on } CP = \text{sq. on } CQ \; (\because CP = CQ),$$
$$\therefore \text{sq. on } PO = \text{sq. on } QO,$$
$$\therefore PO = QO.$$

Ex. 304.—**A is a pt. external to a given ⊙ BCD whose centre is E; from D, the point where AE cuts the ⊙, is drawn DF ⊥r to ED, and meeting the ⊙ AFG described with centre E and radius EA in F; if A be joined to the B where EF cuts the given ⊙ BCD, show that AB is the tangent at B.** (Euclid's own construction.)

Ex. 305.—Show that Euclid's own construction enables us to draw two tangents to a given circle from an external point.

Ex. 306.—**If any number of circles touch two given straight lines the centres all lie on the bisectors of the angles made by the two given lines.** (*See diagram on* p. 51.)

Conversely :—**With any point on a bisector of an angle as centre a circle can be described to touch the straight lines which make the angle.**

Hence :—**The locus of the centre of a circle which touches two intersecting straight lines is the pair of straight lines bisecting the angles between them.**

Ex. 307.—CP and CQ (diag. on p. 192) are tangents to the ⊙ described with centre O, and radius OP or OQ.

When two intersecting circles, as in this exercise, are such that the tangents at either point of intersection are at right angles to each other they are said **to cut each other orthogonally.**

Ex. 308.—With the same construction, CO bisects PQ at right angles.

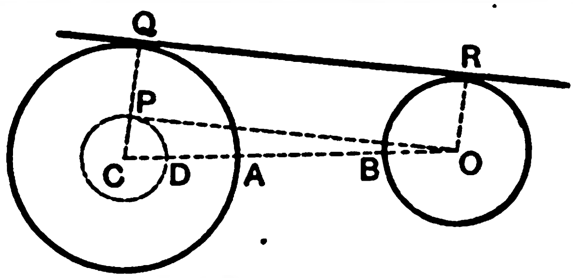
Ex. 309.—**If two circles cut each other orthogonally the sum of the squares of their radii is equal to the square of the line joining their centres.**

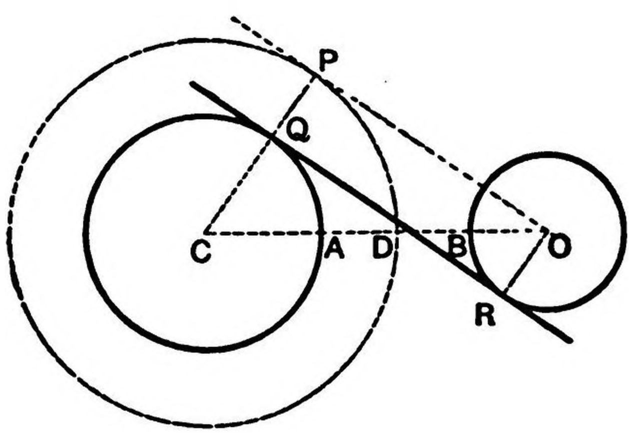
Ex. 310.—Enunciate and prove the converse of Ex. 309.

Ex. 311.—**Through a given point to draw a chord of a circle equal to a given straight line.**

Chords of a given length envelop a certain circle (see Ex. 295). *Consequently a tangent to this circle from the given point will give a chord as required.*

Ex. 312.—**To draw a common tangent to two given circles.**





Let C and O be the centres of the two ⊙s AQF, BRG, A and B the points where CO cuts the ○ces, and suppose $CA > OB$.

From AC or AO cut off AD equal to OB. With centre C and rad. CD describe a ⊙ ; draw a tangent OP to it from O.

Let CP, or CP produced, cut ⊙ AQF in Q. Through O, *on the same side of* OP *as* Q *is*, draw OR ∥ CP, and join QR.

QR shall be a common tangent.

Radii CQ, CP = radii CA, CD ;
∴ PQ = AD
= OB.
= OR.
But OR ∥ PQ.
∴ OPQR is a ∥gm ;
and ∵ OPQ is a rt. ∠,
∴ ∠s at Q and R are also rt. ∠s.
∴ QR touches the ⊙s at Q and R.

The figures are drawn for the cases in which the two given circles do not meet one another. The student should draw figures for all other possible cases (see list on p. 184), and should satisfy himself that the number of common tangents possible to two given circles is (1) 0, (2) 1, (3) 2, (4) 3, (5) 4.

(*The numbers* (1), (2), (3), (4), (5), refer to the *list on* p. 184.)

A common tangent is called **exterior, or interior,** according as it has the two circles on the same side as in (i.), or on opposite sides of it, as in (ii.).

Ex. 313.—If two exterior, or two interior, common tangents intersect, the point of intersection must be on the line through the centres.

For each centre must lie on the line bisecting the angle between the two tangents in which the two circles lie. (See Ex. 306.)

The straight line PQ joining the points of contact of the tangents from O to a circle is often called the **chord of contact** of the tangents from O.

Ex. 314.—**Any two tangents to a circle are equally inclined to their chord of contact.**

Ex. 315.—**If two circles touch each other a common tangent can be drawn at the point of contact.**

Ex. 316.—**Enunciate and prove the converse of** Ex. 315.

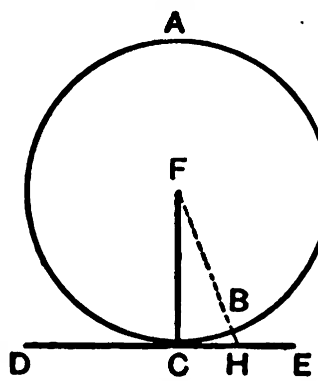
Ex. 317.—**Through the point of contact of two circles which touch each other either internally or externally a straight line is drawn meeting the circumferences again in P and Q; show that the tangents at P and Q are parallel.**

Ex. 318.—**State and prove the converse of** Ex. 317.

PROPOSITION 18. THEOREM.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

Let the st. line DE touch ⊙ ABC at the pt. C, and let the rad. FC be drawn; then FC shall be ⊥r to DE.



If not, draw FH ⊥r to DE, cutting the ○ce at B.

Then ∠ FHC is a rt. ∠;

∴ ∠ FCH < a rt. ∠,

∴ ∠ FHC > ∠ FCH,

∴ FC > FH,

∴ FB > FH,

which is impossible.

Similarly it may be shown that no other st. line through F except FC is ⊥r to DE.

Alternative Proof.—For through C one tangent, and one only, can be drawn to ⊙ ABC, [III. 16, COR.

viz. the ⊥r to FC.

∴ the tangent DE must be this ⊥r.

∴ FC is ⊥r to DE.

NOTE.

The Alternative Proof of III. 18 just given forms an excellent example of the **Rule of Identity** :—

If there is but one A and but one B, then from the fact that A is B it necessarily follows that B is A. (Syllabus.)

Ex. 319.—Show that III. 3 follows from the Corollary to III. 1 by the Rule of Identity.

Ex. 320.—To draw a tangent to a given circle (1) parallel, (2) perpendicular, to a straight line.

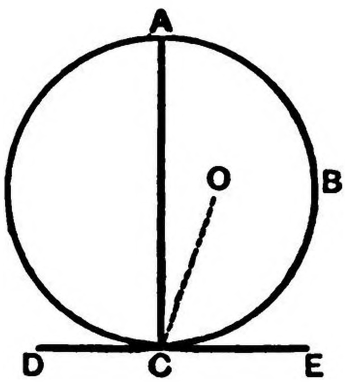
Ex. 321.—Draw a chord of a circle of given length (1) parallel, (2) perpendicular, to a given straight line.

Ex. 322.—The locus of a point such that the sum of the squares of the tangents from it to two given ⊙s is constant is a ⊙. (See Ex. 203).

PROPOSITION 19. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let DE be a tangent to the ⊙ ABC at C, and let CA be drawn ⊥r to DE, then the centre of the ⊙ shall be in CA.



If not, let O outside CA be the centre. Join OC.

Then ∠ OCE is a rt. ∠,

but ∠ ACE is a rt. ∠;

∴ ∠ OCE = ∠ ACE, which is absurd;

∴ O cannot lie outside CA,

i.e. CA must pass through the centre.

Alternative Proof.—It has been shown that if the centre
F be joined with C,

FC is $\perp$r to DE,

$\therefore$ it must fall along CA,

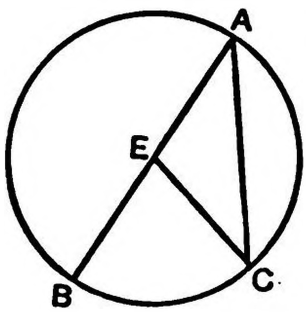
$\therefore$ CA must pass through the centre.

Ex. 323.—Demonstrate III. 19 by the Rule of Identity.

Def.—**An angle is said to 'stand' (or 'insist') upon the circumference intercepted between the straight lines which contain the angle.**

PROPOSITION 20. Theorem.

The angle at the centre of a circle is double of the angle at the circumference upon the same base, *i.e.* upon the same part of the circumference.

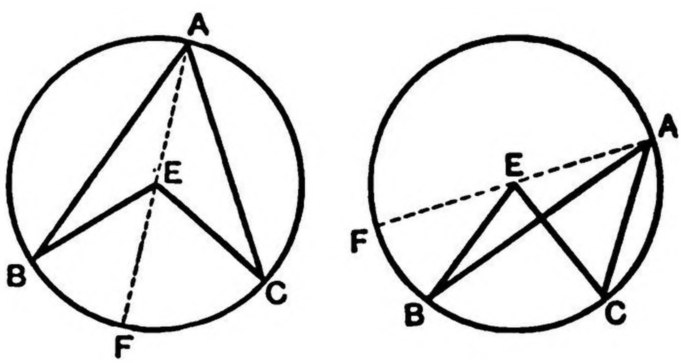


Let ABC be a ⊙, BEC an ∠ at its centre E, and BAC an ∠ at the ○ce, standing on the same arc BC; then ∠ BEC is double of ∠ BAC.

(1) Let one of the sides BA containing the ∠ BAC be a diameter.

Rad. EA = rad. EC,
∴ ∠ EAC = ∠ ECA,
∴ ∠s EAC, ECA are double of ∠ EAC,
∴ ext. ∠ BEC is also double of ∠ EAC.

(2) Let neither of the sides BA, AC be a diameter.
Join AE, and produce it to cut the ○ce in F.



Then by (1) ∠ FEC is double of ∠ FAC,
and ∠ FEB is double of ∠ FAB,
∴ whole or remg. ∠ BEC is double of whole or remg. ∠ BAC.

NOTES.

It will be convenient at this stage to draw the attention of the student to an extension of the idea which he has hitherto attached to the word 'angle.' We shall adopt the treatment of the 'Syllabus.'

When two straight lines are drawn from the same point, they are said to contain or to make with each other, a 'plane angle.'

The point is called the 'vertex,' and the straight lines are called the 'arms' of the angle.

A line drawn from the vertex, and turning about the vertex in the plane of the angle, from the position of coincidence with one arm to the position of coincidence with the other, is said to 'turn through the angle,' and the angle is greater as the quantity of turning is greater.

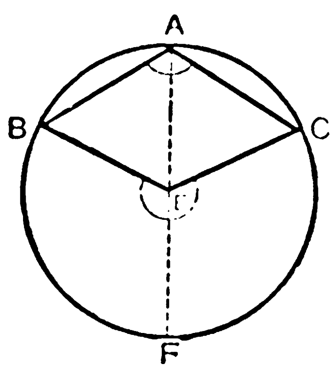
Since the line may turn from the one position to the other in either of two ways, two angles are formed by two straight lines drawn from a point.

These angles (which have a common vertex and common arms) are said to be 'conjugate.'

When the arms are in the same straight line the conjugate angles are equal, and each is then called a straight angle.

When the arms are not in the same straight line, the conjugate angles are not equal. The greater is called the 'major conjugate,' and the smaller the 'minor conjugate' angle. When 'the angle contained by two straight lines' is spoken of, if the conjugate angles are unequal, the 'minor conjugate' angle is to be understood.

N.B.—III. 20 is obviously true for major conjugate angles at the centre as well as for minor conjugates.



The major conjugate angle BEC at the centre is double of the angle BAC at the circumference, *standing on the same arc* BFC.

Ex. 324.—Two chords AEB, CED of a circle ADBC intersect in E. Prove that the angles subtended at the centre by the arcs AC, BD are together double of the angle AEC.

If the chords CA, BD be produced to meet in F, what is the relation between the angles subtended at the centre by the arcs AD, BC and the angle BFC?

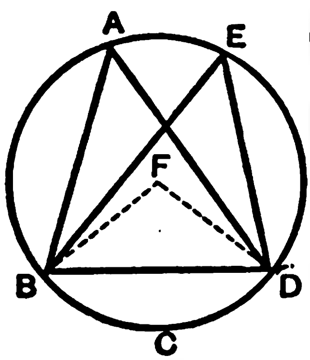
Def.—A segment of a circle is the figure contained by a straight line and the circumference it cuts off.

PROPOSITION 21. Theorem.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a ⊙, and BAD, BED ∠s in the same segment BAED; then ∠BAD = ∠BED.

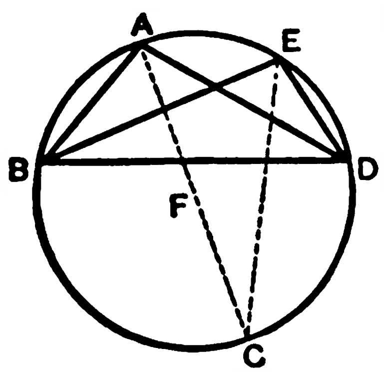
Take F the centre of the ⊙.



(1) Let F fall within the segment BAED and join BF, FD.

Each of the ∠s BAD, BED at the ○ce is equal to half of the ∠ BFD at the centre on the same arc, BCD, [III. 20.

∴ ∠BAD = ∠BED.



(2) Let F not fall within the segment BAED.

Join AF, and produce it to cut the ○ce in C. Join EC.

Then F falls within the segment BADC,

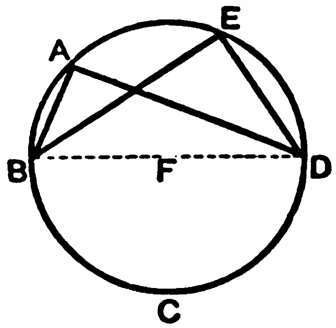
∴ by (1) ∠BAC = ∠BEC.

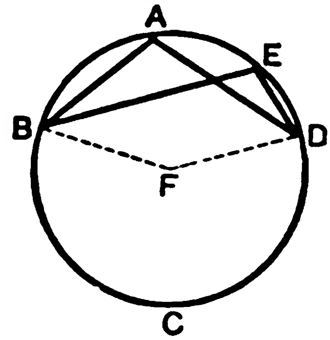
Similarly ∠CAD = ∠CED,

∴ whole ∠BAD = whole ∠BED.

N.B.—If the use of major conjugate angles is admitted, it is plain that the demonstration of III. 21 (1) applies universally, and that (2) is superfluous.

If the student avails himself of this admission, he must be careful to supply the annexed diagrams.





Conversely :—**If a given straight line subtend equal angles at any number of points on the same side of it, an arc can be described with the given straight line as chord which shall pass through all those points.**

For let A and E be two of the pts. at which a given st. line BD subtends equal ∠s.

It could be easily shown that one ⊙, *and one only, could be drawn through* B, A, D.

If this ⊙ does not pass through E, let it cut BE, or BE produced, in E′; then ∠ BE′D = ∠ BAD in the same segment, [III. 21.
= ∠ BED, [Hyp.
which is absurd; [I. 16.
∴ E lies on the ⊙ through B, A, D.

Similarly, all the other points could be shown to lie on the same ⊙.

Hence :—**The locus of a point on one side of a given straight line at which that line subtends a constant angle, is an arc of which that line is the chord** (Syllabus).

Ex. 325.—The angle subtended by the chord of a segment at a point within it is greater than, and the angle subtended at a point outside the segment and on the same side of the chord is less than, the angle in the segment (Syllabus).

Ex. 326.—AB and CD are two parallel chords of a circle ABCD, whose centre is O; AC, BD intersect in E within the circle. Show that a circle can be described to pass through A, E, O, D.

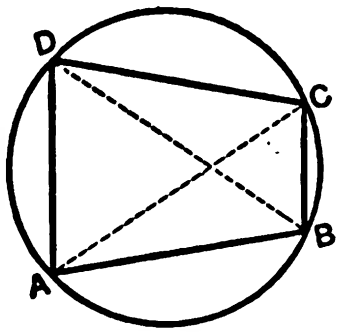
Show angle AED = *twice angle* ABD = *angle* AOD.

Show also that B, E, O, C are concyclic.

PROPOSITION 22. THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral inscribed in the ⊙ ABCD; any two of its opp. ∠s shall together equal two rt. ∠s.
Join AC, BD.



∠ ADB = ∠ ACB in the same segment ADCB, }
and ∠ BDC = ∠ BAC in the same segment BADC, } [III. 21.
∴ whole ∠ ADC = ∠s ACB, BAC,
∴ the two ∠s ADC, ABC = the three ∠s ACB, BAC, ABC,
= two rt. ∠s. [I. 32.
Similarly, the two ∠s BAD, BCD = two rt. ∠s.

Alternative Proof.—Find the centre O of ⊙ ABCD and join AO, OC.
∠ ADC at ○ce = ½ ∠ AOC at centre *on same arc* ABC, }
and ∠ ABC at ○ce = ½ ∠ AOC at centre *on same arc*, ADC, } [III. 20.
∴ ∠s ADC, ABC = ½ sum of the two conjugate ∠s at the centre,
= two right angles.

Conversely:—If a quadrilateral have two of its opposite angles together equal to two right angles, a circle can be described about it.

For let the opposite ∠s BAD, BCD of a quadrilateral ABCD be together equal to two rt. ∠s.

It could easily be shown that one ⊙, and one only, can be described through B, A, D. If this does not pass through C, let it cut BC or BC produced in C′, and join C′D : then ∠ BAD + ∠ BC′D = two rt. ∠s, [III. 21.

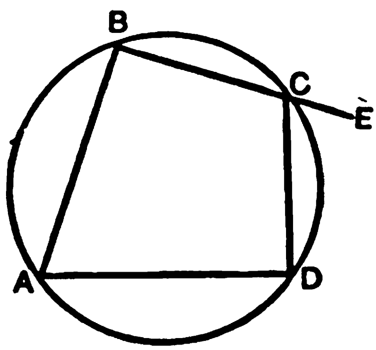
= ∠ BAD + ∠ BCD [Hyp.

∴ ∠ BC′D = ∠ BCD,

which is absurd.

Such a quadrilateral is called a *cyclic quadrilateral.*

Ex. 327.—**If the side BC of a cyclic quadrilateral ABCD be produced to E, then the exterior angle ECD is equal to the interior and opposite angle BAD.** (For each is the supplement of **angle** BCD.)



Conversely :—**If the exterior angle ECD of a quadrilateral ABCD, made by producing BC, be equal to the interior and opposite angle BAD, the quadrilateral is cyclic.**

Prove *indirectly.* See converses of III. 21 and III. 22.

Ex. 328.—Through the two points of section of two intersecting circles are drawn parallel straight lines which are terminated by the circumferences : show that these lines are equal.

Ex. 329.—ABCD is a cyclic quadrilateral. AD and BC being produced meet in E ; prove that triangles ECD, EAB are equiangular.

Note that CD is 'anti-parallel' to AB with respect to E. (See p. 160.)

Ex. 330.—Through a given point to draw a straight line which shall cut off a cyclic quadrilateral from a given triangle.

How many solutions will there be of this problem ?

Ex. 331.—ABC is a triangle : any circle is described to pass through A and B and cut CA, CB in D, E. Show that DE belongs to a set of parallel lines.

Ex. 332.—ABCD is a cyclic quadrilateral: AD, BC are produced to meet in E: a circle described through C and D meets DE, CE produced in F and G. Show that FG is parallel to AB.

Ex. 333.—ABCD is a parallelogram: a circle through AB cuts AD, BC in E and F. Show that a circle can be described through C, D, E, F.

Ex. 334.—The sides AB, CD of the cyclic quadrilateral AB, CD are parallel: E is the mid-point of the arc CD. Show that EC bisects the angle between AC and BC produced. (See Ex. 327.)

Ex. 335.—AB, CD are parallel chords of a circle ABCD, whose centre is O: AC, BD are produced to meet in E. Show that A, E, O, D are concyclic. (Compare Ex. 326.)

Ex. 336.—If a cyclic quadrilateral have two sides parallel, it must have the other two sides equally inclined to the parallels.

State and prove the converse theorem.

Ex. 337.—If a cyclic quadrilateral is a parallelogram, it must be a rectangle.

Conversely:—A rectangle is a cyclic quadrilateral.

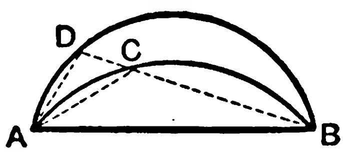
Ex. 338.—If a hexagon ABCDEF be inscribed in a circle, then angles A, C, E together = angles B, D, F.

Def.—Similar segments of circles are those in which the angles are equal, or which contain equal angles.

PROPOSITION 23. Theorem.

Upon the same straight line, and on the same side of it, there can not be two similar segments of circles not coinciding with each other.

Let ADB, ACB be two segments of ⊙s upon the same st. line AB, and not coinciding with each other, then they shall not be similar.



∵ ⊙s ADB, ACB cut at A and B,

∴ they do not cut at any other point. [III. 10.

∴ one segment, ACB, must fall within the other, ADB.

Draw the st. line BCD and join AC, AD.

The ext. ∠ ACB > int. and opp. ∠ ADB,

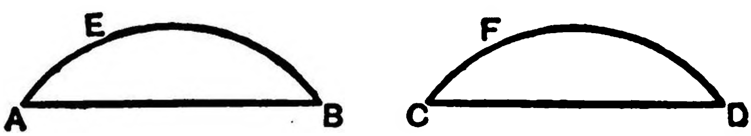
∴ segment ACB is not similar to segment ADB.

See Notes and Exercises on III. 21.

PROPOSITION 24. THEOREM.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be similar segments of ⊙s upon the equal st. lines AB, CD; then segment AEB = segment CFD.



For if the segment AEB be applied to the segment CFD so that the pt. A falls on the pt. C, and the st. line AB along the st. line CD, the pt. B will coincide with the pt. D (∵ AB=CD).

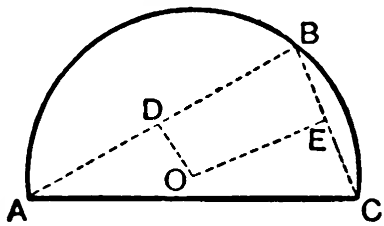
∴ segment AEB must coincide with segment CFD. [III. 23.

∴ segment AEB = segment CFD.

PROPOSITION 25. PROBLEM.

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given segment; it is reqd. to describe the ⊙ of which ABC is a segment.



Take any pt. B on the arc ABC,
and join AB, BC, and bisect AB, BC in D and E.
Through D and E draw ⊥rs to AB, BC.
These ⊥rs must both pass through the centre, [III. 1, Cor.
∴ they must, if produced, intersect.
Let them be produced to intersect in O.
Then O is the centre.
∴ the ⊙ described with centre O and rad. OA, OB, or OC is the ⊙ required.

Note that we have solved the problems—

(1) **An arc of a circle being given, to complete the circle.**

(2) **To find the centre of a circle of which only a part of the circumference is given.**

Ex. 339.—**ABC is a segment of a circle; from the mid-point D of the chord AC is drawn DB perpendicular to AC: at A is made the angle BAE, equal to angle ABD by the straight line AE meeting DB at E. Show that EC = EB = EA.** (*Euclid's own method of finding the centre.*)

Def.—Equal circles are those of which the radii are equal. 'This is not a definition but a theorem, the truth of which is evident; for if the circles be applied to one another so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.' (Simson.)

Ex. 340.—Circles in which equal or supplementary angles are subtended by equal chords are equal.

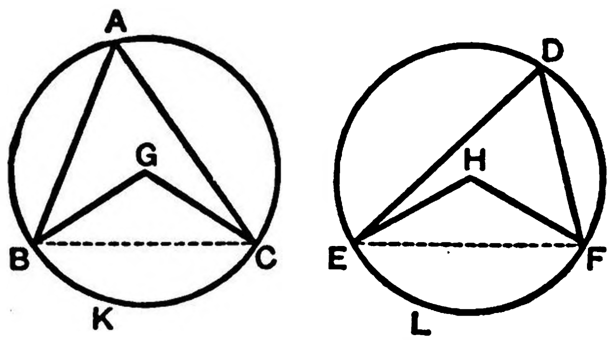
PROPOSITION 26. THEOREM.

In equal circles equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Let G and H be the centres of the two equal ⊙s ABC, DEF.

(1) Let ∠ BGC = ∠ EHF,

then arc BKC = arc ELF.



Apply ⊙ ABC to ⊙ DEF,

so that G falls on H

and GB along HE.

Then arc BKC falls along arc ELF (∵ ⊙ ABC = ⊙ DEF),

and GC falls along HF (∵ ∠ BGC = ∠ EHF),

∴ C coincides with F (∵ ⊙ ABC = ⊙ DEF).

∴ arc BKC coincides with arc ELF,

∴ arc BKC = arc ELF.

(2) Let $\angle$ BAC $= \angle$ EDF,

then $\angle$ BGC $= \angle$ EHF; [III. 20.

$\therefore$ arc BKC $=$ arc ELF.

Corollary.—**In the same circle equal angles stand upon equal arcs, whether they be at the centres or the circumferences.**

For in the $\odot$ ABC let another $\angle$ B′GC′ $= \angle$ BGC.

Then $\angle$ B′GC′ $= \angle$ EHF.

$\therefore$ arc B′C′ on which it stands $=$ arc ELF, [III. 26.

$=$ arc BKC.

This Corollary is very important.

Ex. 341.—If $\odot$ ABC $= \odot$ DEF, but $\angle$ BGC $> \angle$ EHF, prove that arc BKC $>$ arc ELF.

Also if $\angle$ BGC $< \angle$ EHF,

then arc BKC $< \angle$ ELF.

Note the results of III. 26. Its Corollary, and the above Exercises on it, may be thus summarised :—

In the same circle, or in equal circles, equal angles at the centre stand on equal arcs, and of two unequal angles at the centre the greater angle stands on the greater arc. (Syllabus.)

Ex. 342.—If, in a circle ABCD, chord AD is parallel to chord BC, show that arc AB $=$ arc CD. *Use the Corollary.*

Ex. 343.—O is the in-centre of the triangle ABC ; AO is produced to cut the circum-circle of a $\triangle$ ABC in F. Show that FB $=$ FO $=$ FC.

Ex. 344.—(i.) **The internal bisector of any angle at the circumference of a circle bisects the arc on which it stands.**

(ii.) **The external bisector of any angle at the circumference of a circle bisects the arc of the segment in which the angle is.**

Hence :—**The bisectors of the angles in a segment of a circle form two pencils of concurrent lines.**

Note that the *centres* of the pencils (*i.e.* their points of concurrence) are at the ends of the diameter which bisects the segment.

Ex. 345.—Two adjacent sides of a square pass through two fixed points. Show that one of its diagonals passes through another fixed point.

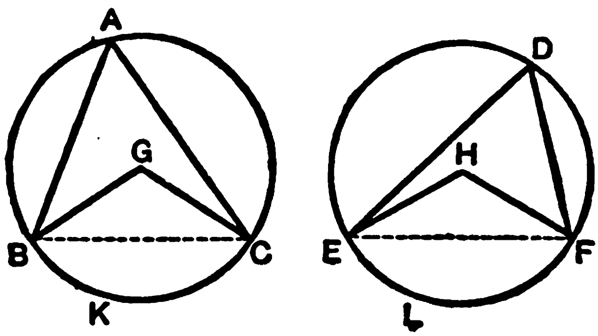
Use Ex. 344 (i.) and the converse of III. 21.

PROPOSITION 27. THEOREM.

In equal circles the angles which stand upon equal arcs are equal to one another, whether they be at the cèntres or at the circumferences.

Let G and H be the centres of the equal ⊙s ABC, DEF, let the arc BKC = arc ELF; then ∠ BGC = ∠ EHF, and ∠ BAC = ∠ EDF.

Apply ⊙ ABC to ⊙ EDF
so that G falls on H,



and GB along HE;
then B falls on E,
and arc BKC along arc ELF (∵ ⊙ ABC = ⊙ DEF).
∴ C coincides with F (∵ arc BKC=arc ELF),
∴ GC coincides with HF, [Ax. 10.
∴ ∠ BGC coincides with ∠ EHF,
∴ ∠ BGC= ∠ EHF,
∴ also ∠ BAC= ∠ EDF. [III. 20.

COROLLARY.—In the same circle the angles which stand upon equal arcs are equal, whether they be at the centres or at the circumferences.

This Corollary is very important.

N.B.—III. 27 can be proved indirectly by means of III. 26.

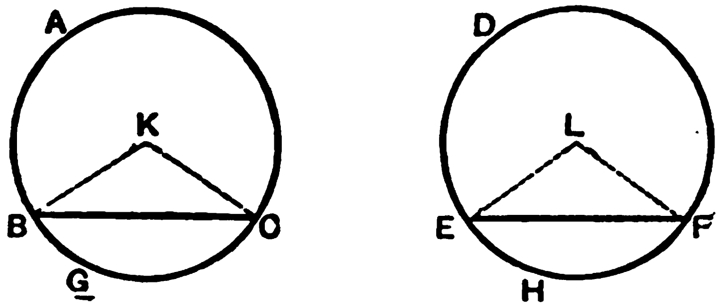
Ex. 346.—If, in a circle ABCD arc AB=arc CD, show that chord AD is parallel to chord BC. *Use the Corollary.*

Ex. 347.—The chord joining the vertex of an angle at the circumference of a circle to the mid-point of the arc on which it stands bisects the angle.

PROPOSITION 28. THEOREM.

In equal circles equal chords cut off equal arcs, the greater equal to the greater and the less to the less.

Let ⊙ ABC = ⊙ DEF,
and chd. BC = chd. EF;
then major arc BAC = major arc EDF,
and minor arc BGC = minor arc EHF.
Find K and L, the centres of ⊙s ABC, DEF;
Join BK, KC, EL, LF.



Apply ⊙ ABC to ⊙ DEF
so that B falls on E,
and BC along EF;
then C coincides with F (∵ BC=EF),
and K coincides with L. [I. 7.
∴ arc BGC coincides with arc EHF, and arc BAC coincides with arc EDF; (∵ ⊙ ABC = ⊙ DEF).
∴ arc BGC = arc EHF,
and arc BAC = arc EDF.

Alternative Proof.—In △s BKC, ELF,
BK, KC = EL, LF (∵ ⊙ ABC = ⊙ DEF),
and BC = EF.
∴ ∠ BKC = ∠ ELF,
∴ arc BGC = arc EHF, [III. 26.
∴ remg. arc BAC = remg. arc EDF (∵ ⊙ ABC = ⊙ DEF).

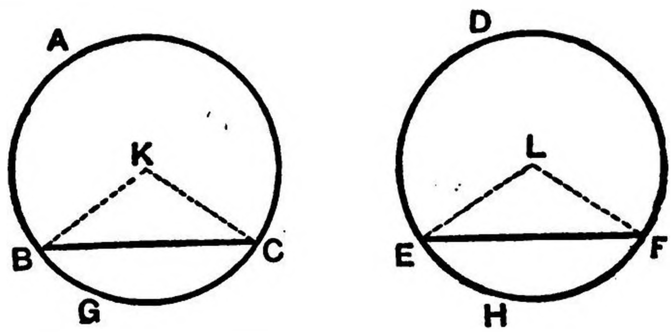
COROLLARY.—**In the same circle equal chords cut off equal arcs.**
This Corollary is very important.

PROPOSITION 29. THEOREM.

In equal circles equal arcs are subtended by equal chords.

Let ⊙ ABC = ⊙ DEF,
and let arc BGC = arc EHF;
then chd. BC = chd. EF.

Find K and L, the centres of ⊙s ABC, DEF, and join BK, KC, EL, LF.



Apply ⊙ ABC to ⊙ EDF,
so that K falls on L,
and KB along LE;
then B falls on E,
and arc BGC along arc EHF (∵ ⊙ ABC = ⊙ DEF),
and C coincides with F (∵ arc BGC = arc EHF).
∴ BC coincides with EF, [Ax. 10.
∴ BC = EF.

COROLLARY.—**In the same circle equal arcs are subtended by equal chords.**

This Corollary is very important.

Ex. 348.—In the same or in equal circles, equal chords subtend equal angles whether they be at the centres or the circumferences.

Ex. 349.—In a circle ABCD, chord AB = chord CD; show that AD is parallel to BC.

Ex. 350.—Demonstrate III. 29 without using superposition.

Ex. 351.—Demonstrate the Corollaries of III. 26, 27, 28, 29 by rotation of one of the two given magnitudes about the centre of the circle.

Ex. 352.—ABCD is a circle having chord AD parallel to chord BC. Show that chord AB = chord CD.

PROPOSITION 30. PROBLEM.

To bisect a given arc.

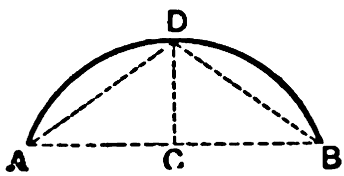
Let ADB be the given arc; it is reqd. to bisect it.

Join AB and bisect it in C; from C draw CD ⊥r to AB to cut the arc in D: then arc ADB is bisected at D.

Join AD, BD.

In △s ACD, BCD,

AC, CD = BC, CD,



and rt. ∠ ACD = rt. ∠ BCD;

∴ AD = DB,

and each of the arcs AD, BD < a semi-⊙

(∵ CD or CD produced passes through the centre),

∴ arc AD = arc DB. [III. 28.

Alternative Construction.—Draw the remg. arc of the ⊙ [III. 25.

and take any pt. E on it. Join AE, EB, and bisect the ∠ AEB by the st. line ED.

Then the given arc is bisected in D,

∵ ∠ AED = ∠ BED in the same ⊙;

∴ arc AD = arc BD. [III. 26.

Ex. 353.—Bisect an arc by bisecting the angle it subtends at the centre.

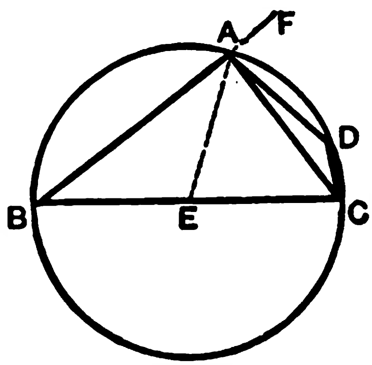
Suggest a reason for Euclid's having chosen the construction given in III. 30 in preference to this.

Ex. 354.—Solve III. 30 by joining any pt. B on the arc ABC to A and C, and drawing the external bisector of the angle ABC. (See Ex. 347.)

PROPOSITION 31. Theorem.

In a circle the angle in a semi-circle is a right angle; but the angle in a segment greater than a semi-circle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a ⊙ of which BC is a diamr. and E the centre, and let CA be drawn dividing the ⊙ into segts. ABC, ADC.



Then (1) ∠ BAC in semi-⊙ BADC is a rt. ∠.
(2) ∠ ABC in major segt. ABC $<$ a rt. ∠.
(3) ∠ ADC in minor segt. ADC $>$ a rt. ∠.

Join EA, and produce BA to F.

(1) Rad. EA = Rad. EB,
∴ ∠ EAB = ∠ EBA.
Similarly ∠ EAC = ∠ ECA,
whole ∠ BAC = ∠s EBA, ECA,
= ext. ∠ CAF, [I. 32.
∴ ∠s BAC, CAF are rt. ∠s.

(2) Int. ∠ ABC $<$ ext. opp. ∠ CAF, [I. 17.
∴ ∠ ABC $<$ a right ∠.

(3) The two ∠s ADC, ABC = two rt. ∠s, [III. 22.
but ABC $<$ a rt. ∠,
∴ ADC $>$ a rt. ∠.

Alternative Proof.

(1) $\angle$ BAC at $\bigcirc$ce $= \frac{1}{2}$ $\angle$ BEC at the centre,

$\therefore$ $\angle$ BAC = a rt. $\angle$.

(2) $\angle$ ABC at $\bigcirc$ce $= \frac{1}{2}$ minor conj. $\angle$ AEC at the centre,

$\therefore$ $\angle$ ABC $<$ a rt. $\angle$.

(3) $\angle$ ADC at $\bigcirc$ce $= \frac{1}{2}$ major conj. $\angle$ AEC at the centre,

$\therefore$ $\angle$ ADC $>$ a rt. $\angle$.

COROLLARY.—**From this it is manifest that if one angle of a triangle be equal to the other two it is a right angle.**

Note that III. 31 (1) and the above Corollary might have been inserted immediately after I. 32.

Note that by the **Rule of Conversion** (see page 185) it follows that :—
A segment of a circle is less than, equal to, or greater than a semi-circle according as the angle in it is greater than, equal to, or less than a right angle.

Ex. 355.—Circles are described on the sides of a triangle as diameters; show that their pts. of intersection all lie on the sides of the triangle.

Ex. 356.—Through one of the points of intersection of two circles diameters are drawn; show that the other ends of the diameters and the other point of intersection are **collinear** (*i.e.* lie in the same straight line).

Ex. 357.—In a right-angled $\triangle$, if a semi-circle be described on one of the sides containing the right angle, the tangent at the points where it cuts the hypotenuse bisects the other side.

Ex. 358.—ABCD is a straight line: circles are described on AB, CD as diameters, and a common tangent is drawn to the circles, the points of contact being E and F. Prove that triangles AEB, CFD are equi-angular. (*Join* E *and* F *to the respective centres.*)

Ex. 359.—**The mid-points of a set of chords of a circle which all pass through the same point all lie on another circle.** (*Join any mid-point with the centre of the given circle and use* III. 3.

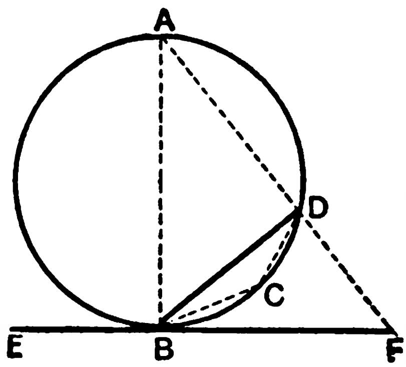
Ex. 360.—Prove III. 31 by Ex. 65.
(*Produce* BA *to* F, *so that* AF = AB, *and join* CF.)

Ex. 361.—To draw through a given point a chord of a given circle which shall be bisected by a given straight line.

PROPOSITION 32. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle shall be equal to the angles in the alternate segments.

Let ABCD be a ⊙; EBF the tangent at B, and BD any chord through B;
then (1) ∠ FBD = ∠ in alt. segt. BAD;
and (2) ∠ EBD = ∠ in alt. segt. BCD.



From B draw BA at rt. ∠s to EF, and ∴ passing through the centre. Join AD, DC, CB. Produce AD to F.

∵ ADB is a semi-⊙
(∴ ∠ ADB is a rt. ∠),
∴ ∠ BDF is a rt. ∠,
∴ ∠ BDF = ∠ ABF.

But ∠ BFD is common to △s BDF, ABF,
∴ 3rd ∠ FBD = 3rd ∠ BAD. [I. 32

(2) Again ∠s FBD, EBD = two rt. ∠s,
= ∠s BAD, BCD, [III. 22.
and ∠ FBD = ∠ BAD,
∴ ∠ EBD = ∠ BCD.

Conversely :—**If through an end of a chord of a circle a straight line be drawn making angles with the chord equal to the angles in the alternate segments, this straight line shall be a tangent to the circle.**

Let BD be a chord of the circle ABCD, and let EBF be drawn through B, making ∠ FBD = ∠ in segment BAD
(and ∴ ∠ EBD = ∠ in segment BCD),
then EBF shall be the tangent at B.

Draw the diamr. BA *through* B.

Join AD and produce it to F.

Then ∠ BAD = ∠ FBD.

But ∠ BFD is common to △s ABF, BDF,

∴ 3rd ∠ ABF=3rd ∠ BDF,

=a rt. ∠;

∴ EBF is the tangent at B.

This converse can also be demonstrated indirectly. It is important for subsequent work.

Ex. 362.—**The tangent at A to the circum-circle of the triangle ABC is parallel to any anti-parallel to BC with respect to A.** (See Ex. 105 and 257.)

Ex. 363.—ABC is a triangle; D and E are taken on AB, AC such that DE is parallel to BC: show that the circum-circles of triangles ABC, ADE have a common tangent at A.

Ex. 364.—ABCD is a cyclic quadrilateral; AD, BC produced meet in E: prove that the tangent at E to the circum-circle of CDE is parallel to AB.

Ex. 365.—Use III. 32 to show that the tangents to a circle from an external point are equal.

Ex. 366.—**Two circles touch each other** (externally or internally); show that :—

(i.) **If through the point of contact any straight line be drawn it will cut off similar segments.** (*Draw the common tangent at the point of contact.*)

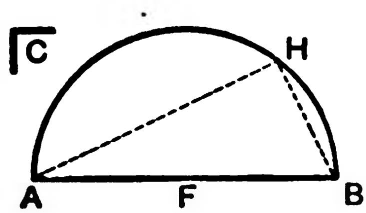
(ii.) **If through the point of contact any straight line be drawn to cut the circles again, the tangents at the other points of section will be parallel.**

(iii.) **If through the point of contact any two straight lines be drawn to cut the circles again, the chords joining the other points of section will be parallel.**

PROPOSITION 33. PROBLEM.

Upon a given straight line to describe a segment of a circle containing an angle equal to a given rectilineal angle.

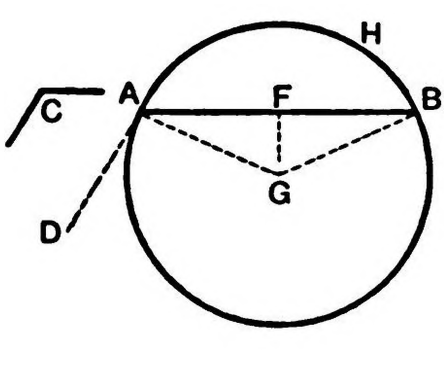
Let AB be the given st. line, and C the given rect. $\angle$: It is

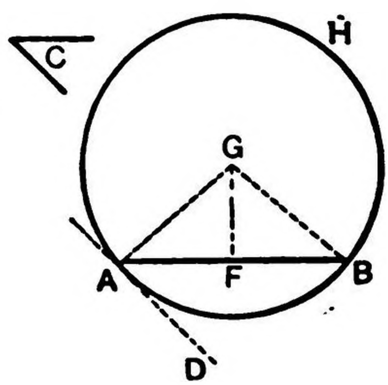


reqd. to describe on AB a segt. of a $\odot$ containing an $\angle$ equal to C. Bisect AB at F.

(1) If C is a rt. $\angle$,
with centre F and rad. FA or FB, describe the semi-$\odot$ AHB.
Then $\angle$ AHB is a rt. $\angle$,
$\therefore$ $\angle$ AHB $=$ $\angle$ C.

(2) If $\angle$ C be not a rt. $\angle$,
at A make $\angle$ BAD equal to C, and draw AG $\perp$r to AD; through F draw FG $\perp$r to AB, and join GB.





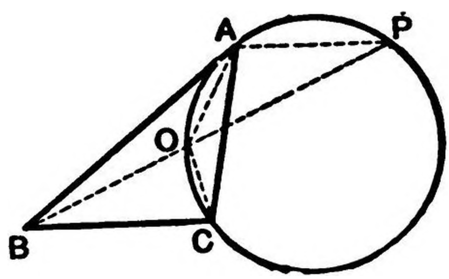
In the $\triangle$s AFG, BFG,
AF, FG $=$ BF, FG,

and rt. ∠ AFG = rt. ∠ BFG,
∴ AG = BG,
∴ ⊙ AHB described with centre G and rad. GA shall pass through B.

Also ∵ GAD is a rt. ∠,
∴ AD is the tangent at A;
∠ BAD = ∠ in alt. segt. AHB,
∴ ∠ in segt. AHB = given ∠ C.

Note that we are shown how to describe a circle which shall touch a given straight line, at a given point, and pass through another given point.

Ex. 367.—**To find a point O within a triangle ABC such that angle OAB = angle OBC = angle OCA.**



Describe a ⊙ passing through C, and touching AB at A.
Draw the chord AP parallel to BC.
Join BP, cutting the ○ce in O.
Then ∠ OAB = ∠ OCA in alt. segt. [III. 32.
= ∠ OPA in same segt. [III. 21.
= alt. ∠ OBC.

Ex. 368.—Find a point O′ within a triangle ABC, such that angle O′BA = angle O′CB = angle O′AC.

These two points O and O′ are called the **Brocard points** of the triangle ABC.

The construction given above we owe to Mr. R. F. Davis, M.A.

Ex. 369.—(i.) Two circles touch internally; prove that the segments of a chord of the outer circle which touches the inner subtend equal circles at the point of contact.

(ii.) Enunciate and prove a similar theorem for two circles touching externally.

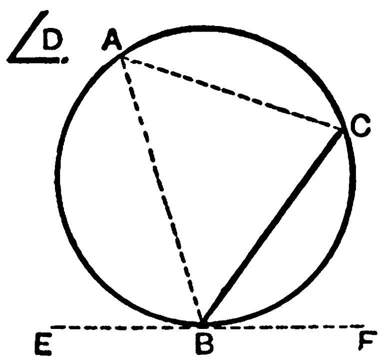
Ex. 369 (*a*).—On a given straight line to describe a segment similar to a given segment.

Ex. 369 (*b*).—In the Fig. of III. 32 show that rect. AF, AD = sq. on AB for all positions of D. (Compare Exx. 479, 480, 481, 592 (*a*)).

PROPOSITION 34. PROBLEM.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given ⊙, and D the given rect. ∠ ; it is reqd. to cut off from ⊙ ABC a segt. that shall contain an ∠ equal to D.



Draw the tangent EBF, and at B make ∠ FBC = D.
Then ∠ in segt. BAC = ∠ FBC, [III. 32.
= ∠ D.

Ex. 370.—Solve the above problem without drawing a tangent.

Ex. 371.—The chord of a given circle is produced: on the whole line so produced describe, with the simplest possible construction, a segment of circle similar to the given one.

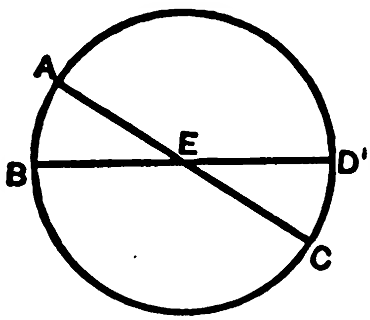
Ex. 371 (*a*).—In the Fig. of III. 34 if BA bisects ∠ EBC, show that A is the mid-point of arc BAC. (Compare Ex. 344).

If also BC bisects ∠ ABF, △ ABC is equilateral.

PROPOSITION 35. THEOREM.

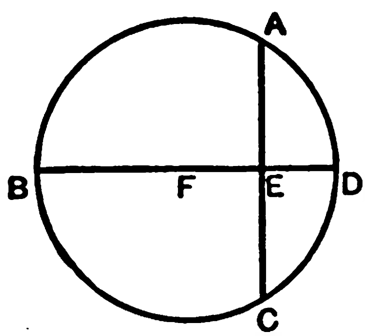
If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two chds. AC, BD of the ⊙ ABCD intersect at E, then rect. AE, EC=rect. BE, ED.



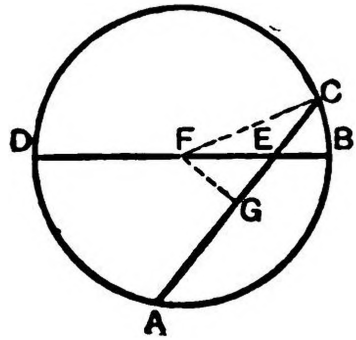
(1) If each passes through the centre, E is the centre, rect. AE, EC = rect. BE, ED (∵ AE = EC = ED = EB).

(2) If one of them, BD, pass through the centre F, and cut the other, AC, which does not pass through the centre, at rt. ∠s,



then AE=EC, [III. 3.
and rect. BE, ED=sq. on AE, [Demonstration of II. 14.
=rect. AE, EC (∵ AE=EC).

(3) If BD pass through the centre F, and cut AC, which does not pass through the centre, but not at rt. ∠s, draw FG ⊥r to AC and join FC.



Then AG=GC,

and rect. BE, ED with sq. on EF= sq. on FB, [II. 5.

= sq. on FC (∵ FB=FC).

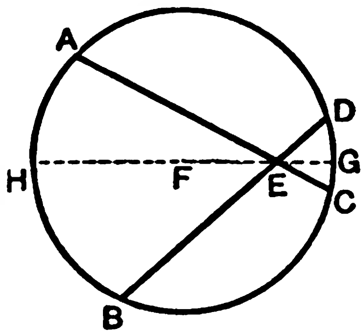
= sqs. on FG, GC, [I. 47.

= sqs. on FG, GE with rect. AE, EC, [II. 5.

But sq. on EF = sqs. on FG, GE, [I. 47.

∴ rect. BE, ED = rect. AE, EC.

(4) If neither pass through the centre F, join EF and produce it to cut the ○ce in H and G.



$$\left.\begin{aligned}\text{Then rect. BE, ED} &= \text{rect. GE, EH,}\\ &= \text{rect. AE, EC,}\end{aligned}\right\}\text{ by cases (2), (3).}$$

Ex. 372.—**Prove the converse of III. 35.** (*Use the indirect method.*)

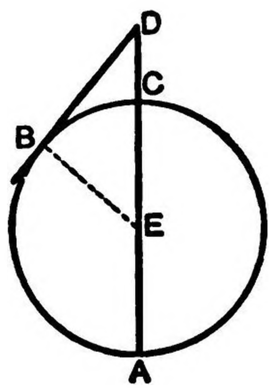
Ex. 373.—ABC is a triangle; AD, BE ⊥r to BC, AC respectively. Cut in O. Show that rectangle AO, OD=rectangle BO, OE.

PROPOSITION 36. THEOREM.

If from a point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it; the rectangle contained by the whole line which cuts the circle and the part of it without the circle shall be equal to the square of the line which touches it.

From any pt. D without the ⊙ ABC, let a tangent DB and a secant DCA be drawn; then rect. AD, DC=sq. on DB.

(1) If DCA passes through the centre E, join EB

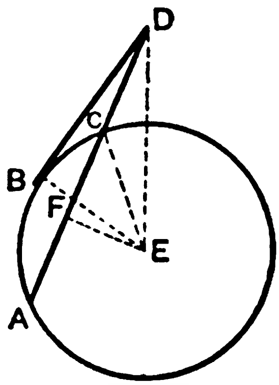


Rect. AD, DC with sq. on EC=sq. on ED, [II. 6.
=sqs. on EB, BD. [I. 47.
But sq. on EC=sq. on EB (∵ EC=EB),
∴ rect. AD, DC=sq. on BD.

(2) If DCA does not pass through the centre E, join EB, EC, ED, and draw EF ⊥r to AC.

Then AF=FC, [III. 3.
∴ rect. AD, DC with sq. on FC=sq. on FD. [II. 6.
∴ rect. AD, DC with sqs. on FC, FE=sqs. on FD, FE,

$\therefore$ rect. AD, DC with sq. on EC = sq. on ED, [I. 47.
= sqs. on EB, BD. [I. 47.



But sq. on EC = sq. on EB ($\because$ EC = EB),
$\therefore$ rect. AD, DC = sq. on BD.

COROLLARY.—(i.) **If from any point without a circle there be drawn two straight lines cutting it, the rectangles contained by the whole lines and the parts of them without the circle are equal to one another.**

For each is equal to the square of the tangent to the circle from the same point. **(See also Ex. 195.)**

COR.—(ii.) **If two tangents to a circle be drawn from the same point, they are equal.**

Ex. 374.—(i.) **If the common chord of two intersecting circles is produced to any point, the tangents to the two circles from this point are equal.**

(ii.) **If the tangents to two intersecting circles from any point be equal, that point must be on the common chord produced.**

Hence :—**The locus of a point from which equal tangents can be drawn to two given intersecting circles consists of the parts external to the circles of the line passing through the points of section.**

For an important extension to this theorem, see page 240 (Radical Axis).

Ex. 375.—**If the common chord of two intersecting circles be produced to cut a common tangent, it will bisect it.**

Ex. 376.—**The three common chords of three circles which intersect each other are concurrent.**

Ex. 377.—Deduce III. 36 from III. 35.

Produce BD, CD to B′, C′, so that DB′ = DB and DC′ = CD, and show that A, B, C′, B′ are concyclic.

Ex. 378.—Deduce III. 36, Cor. (i.), from III. 35.

Ex. 379.—**In equiangular triangles, the rectangles contained by the non-corresponding sides about equal angles are equal.**

Let ABC, AB′C′ *be equiangular triangles, and let them be placed so that* AB′ *falls along* AC, *and therefore* AC′ *along* AB. *Show that* B, C, B′ C′ *are concyclic, and then use* III. 36, Cor. (i.).

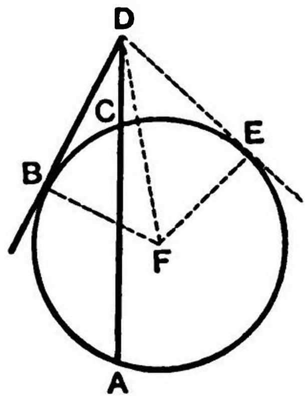
Prove the same theorem also by III. 35.

Ex. 380.—Demonstrate the converse of III. 36, Cor. i.—(i.) indirectly; (ii.) by the converse of III. 35.

PROPOSITION 37. Theorem.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle and the other meets it; if the rectangle contained by the whole straight line which cuts the circle and the part of it without the circle be equal to the square of the line which meets the circle, the line which meets the circle shall touch it.

From a pt. D without the ⊙ ABC, let there be two st. lines, DB, DCA, drawn to the circle, and let rect. AD, DC= sq. on DB, then DB shall touch the circle



Draw the tangent DE.

Find the centre F and join FE, FB, FD.

Then sq. on DB=rect. AD, DC, [Hyp.

=sq. on DE. [III. 36.

∴ DB=DE.

In △s DBF, DEF

DB, BF, FD=DE, EF, FD

∴ ∠ FBD= ∠ FED,

which is a rt. ∠,

∴ FBD is a rt. ∠,

∴ DB is the tangent at B.

Ex. 381.—Prove III. 37 indirectly.

Ex. 382.—CR is a common chord of two circles; P any point on CR produced; MN any chord of either circle which passes through P when produced; PA is a tangent to the other circle. Prove that the circum-circle of triangle AMN touches the latter.

DEFINITIONS.

Book III.

I. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

'This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.'

II. A straight line is said to 'touch' a circle, when it meets the circle, and being produced does not cut it.

III. Circles are said to 'touch' one another, which meet, but do not cut one another.

IV. Straight lines are said to be 'equally distant from the centre' of a circle, when the perpendiculars drawn to them from the centre are equal.

V. And the straight line on which the greater perpendicular falls is said to be 'farther from the centre.'

VI. A 'segment of a circle' is the figure contained by a straight line and the circumference it cuts off.

VII. 'The "angle of a segment" is that which is contained by the straight line and the circumference.'

VIII. An 'angle in a segment' is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.

IX. And an angle is said to insist or 'stand upon' the circumference intercepted between the straight lines that contain the angle.

X. A 'sector' of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.

XI. 'Similar segments' of circles are those in which the angles are equal, or which contain equal angles.

MISCELLANEOUS EXERCISES.—V.

(Book III.)

Ex. 383.—From any point P on a given circle is drawn a straight line PQ equal and parallel to a given finite straight line. Show that the locus of Q consists of two equal circles.

Ex. 384.—Find points P and Q on two given circles respectively such that PQ shall be equal and parallel to a given finite straight line.

Ex. 385.—O is a fixed point; P any point on a given circle. From O is drawn OQ equal to OP and making the angle POQ equal to a given rectilineal angle. Show that Q lies on one of two circles equal to the given one.

Take C *the centre of the given circle and draw* OD *equal to* OC *and making angles* COD *equal to the given rectilineal angle: then* D *is the centre of one of the circles on which* Q *must lie.*

Ex. 386.—A rectangle is formed by drawing through the extremities of each of two chords of a circle at right angles to each other a parallel to the other. Show that its corners lie on a circle concentric with the first.

Ex. 387.—Through a given point draw a circle whose circumference shall be at equal distances from three given points. Vuibert's *Questions de Mathématiques Élémentaires.*

Ex. 388.—AA′, BB′, CC′ are parallel chords of a circle; show that the triangles ABC, A′B′C′ are congruent. (*They are symmetrical with respect to a certain diameter.*)

Ex. 389.—Describe three circles of given radii to touch each other externally.

Ex. 390.—Describe three circles of given radii such that two shall touch each other externally and the third internally.

What condition must be satisfied by the radii if the construction is to be possible?

Ex. 391.—A and B are the centres of two circles CDF, CEG which touch one another at C, DE passes through C; show that AD is parallel to BE.

Ex. 392.—A and B are the centres of two circles CDF, CEG, which touch one another at C: a circle BHL concentric with circle CDF passes through B. If AH passes through D, and DE through C, show that BH is equal and parallel to DE.

Ex. 393.—A and B are the centres of two circles CDF, CEG, which touch one another at C; draw a straight line DE through C equal to a given finite straight line.

What limit is there to the length of the given finite straight line in order that the solution may be possible?

Ex. 394.—Two equal circles cut one another, the centre of each being on the circumference of the other. Show that the square on the common chord is three times the square on the radius.

Ex. 395.—Two equal circles have a common chord AB. If a chord AC of one of them equal to AB when produced passes through the centre of the other, then AB equal radius of either.

Ex. 396.—Describe a triangle having given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex to the base.

Ex. 397.—AB is trisected in C and D; CPD is an equilateral triangle; show that D is the centre of the circum-circle of BPC and AP the tangent at P to the same circle.

Ex. 398.—AB is a diameter and AC a chord of a given circle; the tangents at A and C meet in D; show that $\angle$ ADC = twice $\angle$ BAC.

Ex. 399.—AB is a diameter of a circle, C a given point in AB; find a point in the circumference at which AC, CB will each subtend half a right angle.

Ex. 400.—The sides of a triangle ABC are respectively double of those of the triangle DEF; show that the radius of the circum-circle of triangle ABC is double the radius of the circum-circle of triangle DEF.

Take S, *the centre of the circum-circle of triangle* ABC, *and* d, e, f *the mid-points of* SA, SB, SC, *and show that triangles* def, DEF *are congruent.*

The radius of the circum-circle of a triangle is called its circum-radius.

Ex. 401.—Three circles whose centres are A, B, and C all pass through the same point T, their other points of intersection being p, q, r. If Ap, Bq pass through T, show that Cr also passes through T. Show also that p, q, r, A, B, C are concyclic.

Ex. 402.—On any three straight lines TA, TB, TC, drawn from the same point T, as diameters are described circles which intersect again in P, Q, R. If AP, BQ pass through T, show that CR also passes through T.

(P, Q, R *lie on* BC, CA, AB.)

Ex. 403.—OB′, OC′ are chords of a circle ABC perpendicular respectively to the chords AC, AB. Show that chord BC = chord B′C′.

Ex. 404.—The chords OA′, OB′, OC′ of the circum-circle of a triangle ABC are respectively perpendicular to BC, CA, AB; show that AA′, BB′, CC′ are parallel.

Ex. 405.—If AA′, BB′, CC′ be parallel chords of a circle, show that the chords through A′, B′, C′ respectively perpendicular to BC, CA, AB have a common end.

Show also that the chords through A, B, C respectively perpendicular to B′C′, C′A′, A′B′ have a common end.

Ex. 406.—The locus of a point O from which two tangents OP, OQ can be drawn to a given circle, making the angle POQ equal to a given rectilineal angle, is a concentric circle.

Ex. 407.—**One circle, and only one, can be inscribed in a given triangle.**

Since the locus of the centre of a circle touching two given straight lines drawn from a point is the internal bisector of the angle between them, the cross of two such bisectors is the centre of a circle which can be inscribed in the triangle.

The circle is called the **in-circle**, its centre the **in-centre**, and its radius the **in-radius** of the given triangle.

Ex. 408.—**Four circles, and only four, can be described to touch the straight lines made by producing the three sides of a given triangle indefinitely.**

Since the locus of the centre of a circle which touches two intersecting straight lines is the pair of straight lines bisecting the angles between them (*see second diagram on* p. 51), *the centre of a circle touching the three straight lines must be at the intersections of two such sets of bisectors of angles.*

One of the four circles is the in-circle.

The other three are called **ex-circles**, their centres **ex-centres**, and their radii **ex-radii** of the given triangle.

Ex. 409.—The in-radius of an equilateral triangle is equal to one-third the height of the triangle.

Ex. 410.—The circum-radius of an equilateral triangle is double the in-radius.

Ex. 411.—Each ex-radius of an equilateral triangle is equal to three times the in-radius.

Ex. 412.—If a quadrilateral can have a circle inscribed in it, the sums of opposite sides are equal.

Ex. 413.—If a convex quadrilateral has the sums of opposite sides equal, a circle can be inscribed in it.

For let E be the centre of the ⊙ described touching the three sides AB, BC, CD of a quadl. ABCD in F, G, H.

Draw EK ⊥r to AD, and join FK.

If EK > radius of ⊙,

$\angle$EFK > EKF,

∴ $\angle$AFK < $\angle$AKF;

∴ AK < AF.

Similarly, KD < DH,

and ∴ AD < AF, DH.

But BC = FB, HC.

Ex. 414.—To describe a circle touching three straight lines, two of which, but not all, are parallel.

Ex. 415.—The join of two of the ex-centres of an isosceles triangle is parallel to the base.

Ex. 416.—ABCD is a cyclic quadrilateral, having its diagonals AC, BD at right angles to one another.

Show that arc AB + arc CD = arc BC + arc DA.

Through B *draw a chord* BK *parallel to* CD.

Ex. 417.—Chords AB, DC of a circle when produced intersect at right angles in O. Show that arc DA = arc AC + arc BD + arc BC.

Through B *draw a chord* BK *parallel to* CD.

Ex. 418.—E, F, G, H are the mid-points of the arcs AB, BC, CD, DA of a circle ABCD. Show that EG is perpendicular to FH.

Ex. 419.—Bb, Cc are diameters of the circum-circle of triangle ABC. On the perpr. AP from A to BC is taken T, such that AT = Bc = Cb.

Show that BT, CT are perpendicular to CA, AB.

Hence the existence of the ortho-centre.

Show that AbCT, BcAT *are parallelograms.*

Ex. 420.—The perpendiculars BQ, CR to CA, AB the sides of the triangle ABC intersect in T, and AT is produced to cut BC in P. Show that either of the quadrilaterals BPTR, CPTQ is cyclic, and hence that AT is perpendicular to BC.

Hence the existence of the ortho-centre.

$\angle$ATR = $\angle$AQR in same segment of circle through AQTR.

= int. and opp. $\angle$ of cyclic quadrilateral BRQC.

Ex. 421.—The chord Ap of the circum-circle of triangle ABC cuts BC at right angles at P. On PA is taken a point T such that PT = Pp. Show that BT, CT are perpendicular respectively to CA, AB.

Hence the existence of the ortho-centre.

Show that $\angle$BAP = $\angle$BCp = $\angle$BCT.

Ex. 422.—If two equal circles intersect, each is the locus of the ortho-centre of triangles inscribed in the other on the common chord as base.

Ex. 423.—Three equal circles intersect at a point T, their other points of intersection being A, B, C. Show that T is the ortho-centre of triangle ABC.

Show also that the triangle formed by joining the centres of the circles is congruent with ABC.

Ex. 424.—Two triangles on the same base and on the same side of it, have equal vertical angles. Show that the join of their vertices is parallel to the join of their orthocentres.

Ex. 425.—Two triangles on the same base, and on the opposite sides of it, have their vertical angles supplementary. Show that the join of their vertices is parallel to the join of their ortho-centres.

Ex. 426.—T is the ortho-centre of a triangle ABC whose base BC and vertical angle are given. Show AT is of constant length.

In Exx. 427-439 the following triangle notation is adopted :—

A, B, C vertices.
D, E, F mid points of BC, CA, AB.
P, Q, R projection of A, B, C on BC, CA, AB.
S circum-centre.
T ortho-centre.
H mid-point of ST.
p, q, r where AP, BQ, CR meet circum-circle.
U, V, W mid-points of AT, BT, CT.
Aa, Bb, Cc diameters of circum-circle.

(See *Educational Times*, September 1885.)

PQR is sometimes called the pedal or orthocentric triangle, and DEF the medial triangle, of the triangle ABC.

Ex. 427.—AbCT, BcAT, CaBT are parallelograms.

Ex. 428.—Ta, Tb, Tc pass through, and are bisected at, D, E, F.

Ex. 429.—Tp, Tq, Tr are bisected at P, Q, R.

Ex. 430.—AU = SD = UT.

Ex. 431.—DU = EV = FW.

(*Each = radius of circum-circle of* ABC.)

Ex. 432.—DU, EV, FW pass through, and are bisected at, H, and HU = HV = HW.

Ex. 433.—HP = HQ = HR.

Ex. 434.—A circle passes through D, E, F, P, Q, R, U, V, W.

(This is called the nine point circle of triangle ABC: its centre H is called the mid-centre of triangle ABC.)

Ex. 435.—**AD, ST intersect at a point G, such that DG=⅓ AD, SG=⅓ ST, and HG=⅓ HS.**

Hence the join of the ortho- and circum-centres passes through the centroid and mid-centre.

Ex. 436.—**The circum-circles of triangles ABC, BCT, CAT, ABT, are equal.**

Ex. 437.—**T is the in-centre of triangle PQR.**

Ex. 438.—**A, B, C are the ex-centres of PQR.**

Ex. 439.—$\angle$ ABS = $\angle$ TBC.
$\angle$ BCS = $\angle$ TCA.
$\angle$ CAS = $\angle$ TAB.

Note that the lines joining S and T to any vertex of the triangle ABC are equally inclined to the internal bisector of the angle at that vertex.

Any two lines like AS, AT equally inclined to the bisector of an angle BAC are called with reference to that angle **isogonal lines.**

Any two points S and T, such that the lines joining them to each angle of a triangle are isogonal with reference to that angle, are called **inverse points** with reference to the triangle.

Thus the ortho-centre and circum-centre of a triangle are inverse points.

Ex. 440.—One angle of a triangle exceeds another by a right angle. Show that the tangent at one of its vertices to the circum-circle of the triangle is perpendicular to the opposite side.

Ex. 441.—ACB, ADB are two intersecting circles; the tangents at C and D meet in E. Show that B, C, D, E are concyclic.

Join AB *and use* III. 32.

Ex. 442.—ACB, ADB are two intersecting circles; AC, AD are tangents to ADB, ACB at A. Show that $\angle$ ABC = $\angle$ ABD.

Use III. 32.

Ex. 443.—AB, BC, CD are three equal chords of a circle ABCD. Show that AB, CD are tangents to the circle passing through B, C and the centre of circle ABCD.

Ex. 444.—AB, AC are tangents to a circle BCD. AB, AC are produced to E and F, so that BE=BC=CF. Show that E, B, C, F lie on a circle whose centre is on circle BCD.

Ex. 445.—Two chords of a given circle intersect at right angles at a given point: show that the sum of the squares on the chords is constant.

Ex. 446.—If the diagonals AC, BD of a cyclic quadrilateral intersect at right angles at a fixed point P. The mid-points of AB, BC, CD,

lie on a fixed circle whose centre is half-way between P and that of the given circle.

Ex. 447.—Let PA, PD be two straight lines of given length inclined at any angle; in PA take a point B; find a point C in PD, or PD produced such that rectangle AP, PB shall be equal to rectangle CP, PD. How could you tell by merely considering the angles PAD and PDB whether C falls in PD, at D, or in PD produced?

Ex. 448.—Let two circles touch internally at A, and let the radius of the one be equal to the diameter of the other; draw AB, the diameter of the larger, through A, and BP to touch the smaller circle in P; join AP; show that the square on BP is three times the square on AP.

Ex. 449.—Use III. 15 and III. 35 to show that of all equal rectangles the square has the smallest perimeter.

Ex. 449 (*a*).—Given a vertex, the circum-centre and the ortho-centre to describe the △.

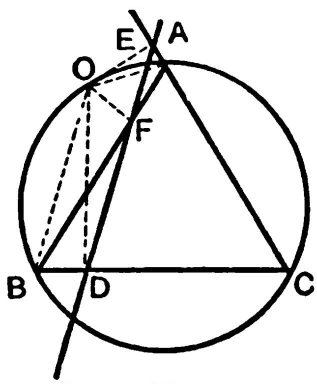
Ex. 449 (*b*).—Any number of △s can be described having the same circum ⊙ and the same ortho-centre.

ON SIMSON'S LINE.

The projections on the sides of a triangle of any point on its circum-circle are in the same straight line.

This straight line is called the '**Simson's line**' or '**Pedal line**' of the triangle with respect to the given point.

Let O be any point on the circum-circle of triangle ABC, OD, OE, OF perpendiculars from O to BC, CA, AB ; then D, E, F are collinear.



Join OA, OB.
Then ∵ ∠s OFB, ODB are rt. ∠s
the ⊙ on OB as diamr. would pass through D and F,
Simy. the ⊙ on OA as diamr. would pass through E and F ;
hence ∠ OFE = ∠ OAE in same segt. of ⊙ through O, E, A, F,
= int. and opp. ∠ OBC of cyclic quadl. OBCA;
∴ ∠s OFE, OFD = ∠s OFD, OBD
= 2 rt. ∠s.

Conversely :—**If the projections on the sides of a triangle of a point be in a straight line, that point is on the circum-circle of the triangle.**

Let the feet D, E, F of the perpr. OD, OE, OF on the sides BC, CA, AB of a △ ABC be collinear.

Then O is on the circum ⊙ of △ ABC.

With the same construction, the ⊙ on OB as diamr. passes through D and F,
and that on OA as diamr. through E and F.
Hence ∠ OBD = ext. ∠ OFE of cyclic quad. OBDF,
= ∠ OAE in same segt. of ⊙ through O, E, A, F ;
∴ O, B, C, A are concyclic.

Note that slight modifications of the diagram and demonstrations may be required when O takes a different position with respect to the triangle.

Ex. 450.—If the projections of any vertex of a quadrilateral on the sides and diagonal of the quadrilateral on which it does not lie be collinear, so will also the projections of any other vertex of the quadrilateral on the sides and diagonal on which it does not lie.

Ex. 451.—**From a point O on the circum-circle of a triangle ABC, any three straight lines OD, OE, OF are drawn, making equal angles with BC, CA, AB,** and *in the same sense.*[1] **Then D, E, F shall be collinear.**

It can easily be shown O, B, D, F are concyclic, and that O, A, E, F are concyclic.

And the demonstration then proceeds as when OD, OE, OF were perpendicular to BC, CA, AB.

Ex. 452.—Enunciate and prove the converse of Ex. 451.

Ex. 453.—**Find a point whose projections on four given intersecting straight lines shall be collinear.**

Ex. 454.—Prove the theorem demonstrated on p. 248 by means of the two theorems just demonstrated on p. 238.

Ex. 455.—Show that the Simson's line of O with respect to ABC, and that of A with respect to OBC, are equally inclined to BC.

Generalise this theorem.

Ex. 456.—**If the perpendiculars from O to BC, CA, AB meet the circum-circle of ABC in p, q, r, the Simson's line of O will be parallel to each of the lines Ap, Bq, Cr.**

Ex. 457.—The Simson-line of a point with respect to an equilateral triangle bisects the radius of the circum-circle drawn to the point.

Ex. 458.—**The Simson-line of O bisects the join of O and the ortho-centre of triangle ABC.**

Ex. 459.—**Circles are described on any three chords OA, OB, OC of a circle as diameters. Shew that their other three points of intersection are in a straight line.**

[1] *I.e.* all to the right or all to the left of the lines from O.

ON THE RADICAL AXIS OF TWO CIRCLES.

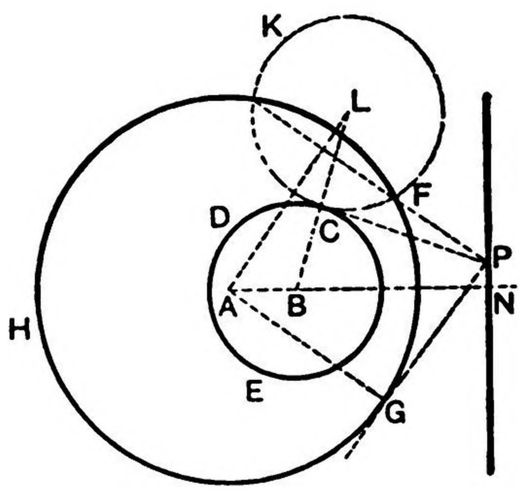
It has already been stated that there exists an indefinite number of points from which equal tangents can be drawn to two given circles which cut one another, and that all these points lie on a certain straight line (see Ex. 374). This straight line is called the **radical axis** of the two circles.

We shall now show that the same theorem holds true *in general* for any two circles, whether they intersect or not; *i.e.* we shall show that—

***In general* a certain straight line can be drawn such that equal tangents can be drawn to two given circles from an indefinite number of points on it.**

Let A and B be the centres of any two circles FGH, CDE.

At any point C on circle CDE not on the line through A, B draw the tangent CP.



Describe a circle CFK touching CP at C and passing through any point F on circle FGH. [See III. 33.

Let L be its centre.

Then BL passes through C and is at right angles to CP, and AL is at right angles to the common chord or common tangent through F to circles CFK, FGH.

Let this chord or tangent be produced to meet CP at P. Draw PN perpendicular to line through A, B, and PG to touch circle FGH. Join AG.

(1) Then sq. on CP = sq. of tangt. from P to CFK,
= sq. of tangt. from P to FGH.
(∵ P is on radical axis of circles CFK, FGH.)

(2) Sqs. on AN, NP = sq. on AP,
= sqs. on AG, GP.
Simy. sqs. on BN, NP = sqs. on BC, CP;
∴ $AN^2 - BN^2 = AG^2 - BC^2$;
∴ N is a fixed point,
and ∴ P lies on a fixed st. line ⊥r to AB.

(3) ∵ C may be taken at an indefinite number of points on circle CDE,
∴ an indefinite number of pts. can be found on this st. line, from which equal tangents can be drawn to ⊙s CDE, FGH.

The construction will be found to fail if the two circles CDE, FGH are concentric.

For an elegant investigation of the properties of the radical axis, see Dr. Casey's *Sequel to Euclid.*

Ex. 460.—To draw the radical axis of two given circles.

(*Use two common tangents.*)

Ex. 461.—**The radical axis of any three circles taken in pairs meet in a point.**

(*Use Ex.* 154.)

(*This point is called the* **radical centre** *of the three circles.*)

Ex. 462.—The ortho-centre T of the triangle ABC is the radical centre of the circles described—

(1) On the sides of ABC as diameters;

(2) TA, TB, TC as diameters.

(Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire.*)

Ex. 463.—**On the sides BC, CA, AB of a triangle ABC are taken respectively the three pairs of points D, D′; E, E′; F, F′. If the quadrilaterals EE′FF′, FF′DD′, DD′EE′ are cyclic, then all six points D, D′, E, E′, F, F′ are concyclic.**

Since E, E′, F, F′ are concyclic, rect. AE.AE′ = rect. AF.AF′;
∴ if ⊙s about FF′DD′, DD′EE′ are not coincident,
A is on their radical axis,
which is impossible.

We owe this theorem and the condensed demonstration to Mr. R. F. Davis, M.A.

It may also be demonstrated by using Ex. 154 to show that the perpendicular bisectors of DD′, EE′, FF′ are concurrent.

Ex. 464.—**If three circles do not all intersect, a circle can be described to cut them orthogonally** (see p. 194, and Ex. 421.)

(*Its centre will be the radical centre of the three circles.*)

Ex. 465.—**The difference between the squares of the tangents to two circles from a given point is equal to twice the rectangle contained by the join of their centres and the distance of the given point from the radical axis.**

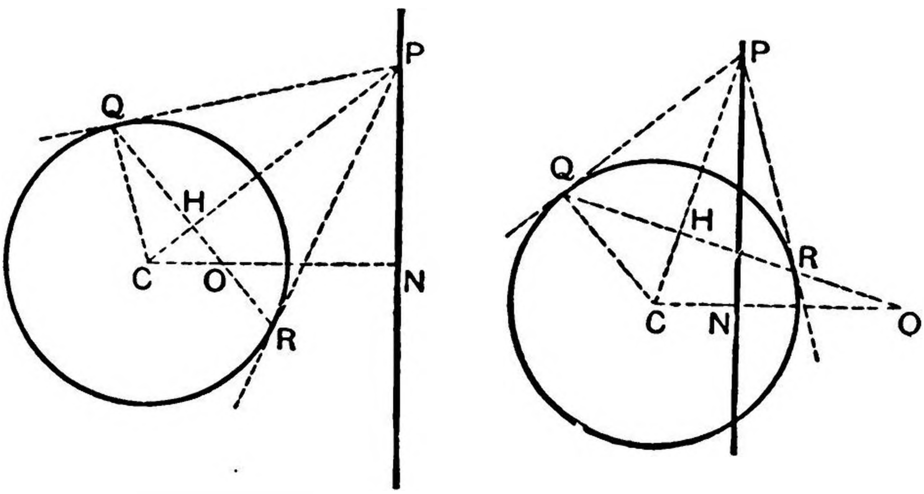
For further properties of Radical Axis or Orthogonal Section, see Casey's *Sequel to Euclid*; M'Dowell's *Exercises on Euclid*; Townsend's *Chapters on the Modern Geometry of the Point, Line, and Circle.*

Ex. 466.—**All circles which cut two given circles orthogonally pass through two fixed points on the line through the centres of the two circles.**

Ex. 467.—Any two circles being given, to find any number of others, all having the same radical axis.

ON POLES AND POLARS.

(1) Let a st. line passing through the fixed point O cut a given $\odot$ whose centre is C in Q and R, and let the tangents at Q and R intersect at P; then P lies on a fixed st. line.



Draw a $\perp$r PN to the line through C, O, and let CP cut QR in H.

Then CP bisects QR at rt. $\angle$s.

$\because$ $\angle$s at H and N are rt. $\angle$s.

$\therefore$ O, H, N, P are concyclic.

$\therefore$ CN.CO $=$ CP.CH. [III. 36.

$= \text{CQ}^2$. [I. 47. See note, p. 90.

But CO is given and CQ is constant.

$\therefore$ N is a fixed point,

$\therefore$ P lies on a fixed st. line.

If QR passed through C, the tangents at Q and R would be $\parallel$ to this fixed st. line.

(2) Let tangents PQ, PR be drawn from any pt. P on a fixed st. line PN to a given $\odot$ whose centre is C, then the line through Q and R shall pass through a fixed point.

Draw CN $\perp$r to NP, and let the line through QR meet CN and CP in O and H.

Then as in (1) CN.CO $= \text{CQ}^2$.

But CN is given and CQ is constant;

$\therefore$ O is a fixed point.

QR is called the **polar** of P, and P the **pole** of QR.

Hence the following general theorems and definitions—

(1) **If tangents be drawn to a given circle at its points of intersection with any straight line passing through a fixed point, they will intersect on a fixed straight line or be parallel to it.**

This fixed straight line is called **the 'polar' of the fixed point with respect to the given circle.**

(2) Conversely :—**If tangents to a given circle be drawn from any point on a fixed straight line, the straight line through the points of contact will pass through a fixed point.**

This fixed point is called **the 'pole' of the fixed straight line with respect to the given circle.**

Note that

The rectangle contained by the distances from the centre of a given circle of a point and its polar with respect to the circle (or of a line and its pole with respect to the circle) is equal to the square of the radius.

This property is sometimes taken as the fundamental one, and used to define the terms **pole** and **polar**; thus,

If upon any straight line through the centre of a given circle, two points be taken on the same side of the centre such that the rectangles contained by their distances from it is equal to the square of the radius, then the perpendicular to the straight line through either of these points is called the 'polar' of the other, which is called the 'pole' of the perpendicular. (See Casey, *Sequel to Euclid.*)

Ex. 468.—**If a straight line (QR) passes through the pole of a second straight line (NP), then the second straight line (NP) passes through the pole of the first.**

Ex. 469.—**If a point (O) lie on the polar of a second point (P), then the second point (P) lies on the polar of the first (O).**

Use the property pointed out in Note (2).

Ex. 470.—**If a point lies on a given straight line, its polar passes through a fixed point.**

(*It passes through the pole of the given straight line.*)

Ex. 471.—**If a line pass through a given point, its pole lies on a fixed straight line.**

(*It lies on the polar of the given point.*)

Ex. 472.—**The point of intersection of the polars of two given points is the pole of the straight line passing through them.**

Or, more briefly,

The cross of the polars of two given points is the pole of their join.

Ex. 473.—**The line joining the poles of two given straight lines is the polar of their point of intersection.**

Or, more briefly.

The join of the poles of two given straight lines is the polar of their cross.

Ex. 474.—ABC is an acute-angled triangle; BQ, CR the perpendiculars on the sides CA, AB. Show that a circle can be described with centre A, such that BQ, CR are the polars of C and B respectively, and hence deduce the existence of the ortho-centre.

(*Rect.* AB.AR = *rect.* AC.AQ. *If* T *be the intersection of* BQ, CR, *it must be the pole of* BC, Ex. 472;

$\therefore$ AT *is* $\perp$*r to* BC.)

ON THE METHOD OF LIMITS.

Def.—If a secant of a circle alters its position in such a manner that the two points of intersection approach and ultimately coincide with one another, the secant in its limiting position is said to 'touch,' or to be a 'tangent' to, the circle (Syllabus).

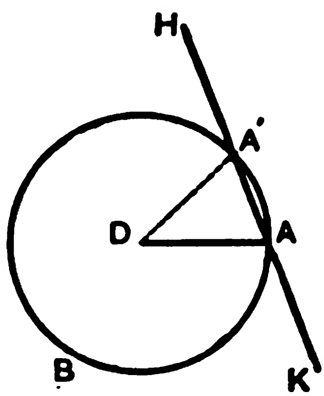
Def.—The point in which the two points of intersection ultimately coincide is called the 'point of contact,' and the tangent is said to 'touch' the circle at that point (Syllabus).

These definitions of tangency will be found to lead to the same results as Euclid's.

PROPOSITION.

The tangent at any point to a circle is at right angles to the radius drawn to the point of contact.

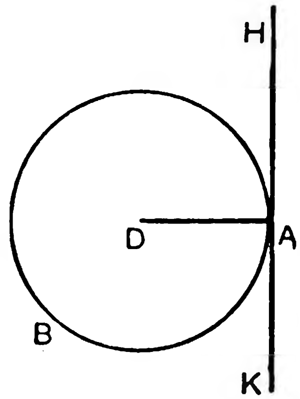
Let D be the centre of the ⊙ AA'B and HA'AK the secant through A' and A. Join DA', DA.



Then ext. ∠ DA'H = ∠s HAD, ADA'
= ∠s KA'D, ADA'
= ext. ∠ DAK.

Now let A' move up to and coincide with A.

Then in the limit when HK becomes the tangent at A, we have
$\angle DAH = \angle DAK$.



$\therefore$ DA is $\perp$r to HK.

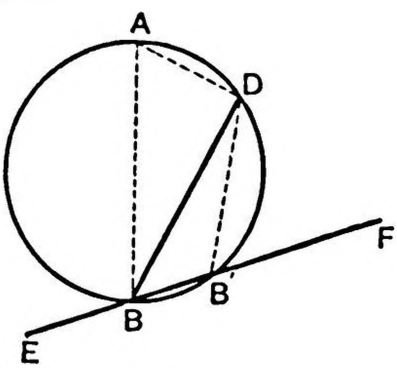
Ex. 475.—Prove the same theorem by using III. 3, and taking the limit.

When using this method, the student should be careful to draw two diagrams, one illustrating what happens before, the other what happens at the coincidence of the two points which approach one another.

PROPOSITION.

Each angle contained by a tangent and a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle. [III. 33.

Let EBB′F be the secant of ⊙ ABB′D through B, B′, and BD any other chd. through B.



$\angle FB'D + \angle EB'D = 2$ rt. $\angle$s.
$= \angle BAD + \angle EB'D$. [III. 22.
$\therefore \angle FB'D = \angle BAD$.

Now let B′ move up to and coincide with B.

Then in the limit, when EBF becomes the tangent at B,
$\angle FBD = \angle BAD$. [See diagram of III. 33.

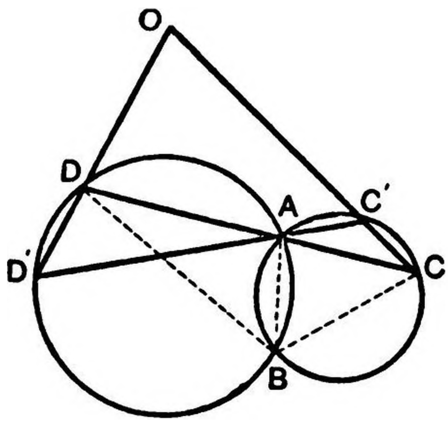
From a given General Theorem, the student may often deduce a special case by the method of limits, of the truth of which he may afterwards satisfy himself by a demonstration which does not depend on that method.

The circum-circles of the four triangles formed by four intersecting straight lines all pass through one point.

Let ADD′, ACC′, OCD, OC′D′ be four △s formed by the four intersecting st. lines OC, OD, CD, C′D′.

Then the circum-⊙s of △s ADD′, ACC′ meet in some other pt. B.

[See Ex. 366 (iii.).



Join BD, BA, BC.

∠ABD = ∠OD′A in same segt. ABD′D,

and ∠ABC = ext. ∠AC′O of cyclic quadl. ABCC′.

∴ ∠CBD = ∠s OD′A, AC′O.

∴ ∠s CBD, COD = ∠s OD′A, OC′A, COD,

= two rt. ∠s.

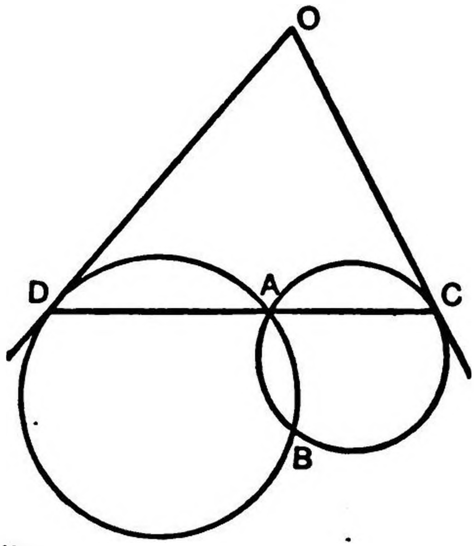
∴ the circum-⊙ of OCD passes through B.

Similarly the circum-⊙ OC′D′ passes through B.

Now, let C′ move up to and coincide with C,

and ∴ D′ move up to and coincide with D.

Then in the limit when OC, OD become the tangents at C and D the circum-⊙ of OCD passes through B.



Enunciate generally.

Again :—

If two circles intersect, and through one of the common points two straight lines be drawn and terminated each way by the circumference, they subtend equal angles at the other common point.

Let ABC, ABD be the two ⊙s, CAD, C′AD′ the two st. lines through A. Join AB, DD′, CC; produce DD′, CC′ to meet in O. (See p. 248.) Then it has been shown that

$$\angle\text{s } CBD,\ COD = 2 \text{ rt. } \angle\text{s}.$$

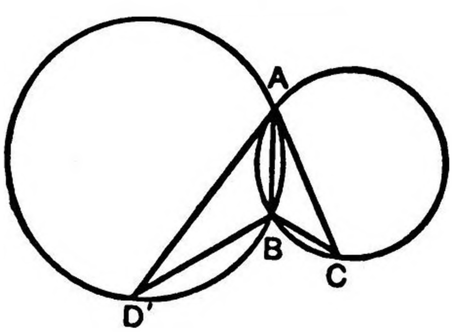
Similarly $\angle$s C′BD′, C′OD = 2 rt. $\angle$s.

$$\therefore \angle CBD = \angle C'BD.$$

Now let C′ and D each move up to and coincide with A.

Then in the limit when CA, D′A become tangents at A we have

$$\angle ABC = \angle ABD'.$$



Hence :—

If two circles intersect, and through one of the common points

a chord be drawn to each circle touching the other, these chords subtend equal angles at the other common point. [See Ex. 369.

Ex. 476.—Prove the first Corollary to III. 36 without assuming III. 36, and then deduce III. 36 by the method of limits.

Prove each rectangle = diffce. between sq. on ED, *and sq. on radius.*

The Method of Limits can also be applied to the contact of circles with one another.

Ex. 477.—Deduce III. 11 and III. 12 from the theorem that

The straight line through the centres of two intersecting circles bisects their common chord at right angles.

Ex. 478.—In the figure on p. 248.

$$\angle CBD = \angle s\ AD'D,\ ACO.$$

Hence show by the Method of Limits that when the two circles touch at A, DD′, CC′ are parallel.

Ex. 478 (*a*).—FA, FB are two tangents to a ⊙ABH. Show by method of limits that ∠AFB is equal to half the difference of the angles at the centre subtended by the two arcs into which AB divides the ⊙.

See Ex. 324.

ON INVERSION.

Def.—If through a fixed point O a straight line is drawn to any point P, and on it is taken a point P′ such that OP, OP′ = the square on a given fixed line K, then the point P′ is called the 'inverse' of the point P.

Def.—If the point P lie on any given line the locus of P′ when taken always along OP, or always in the opposite direction, is called the inverse of the given line.

Def.—The area of the rectangle OP, OP′ is called the 'constant or 'modulus of inversion.'

If a square be described equal to the constant of inversion its side is called the '**radius of inversion.**'

Def.—The point O is called the 'origin,' or 'pole,' of inversion.

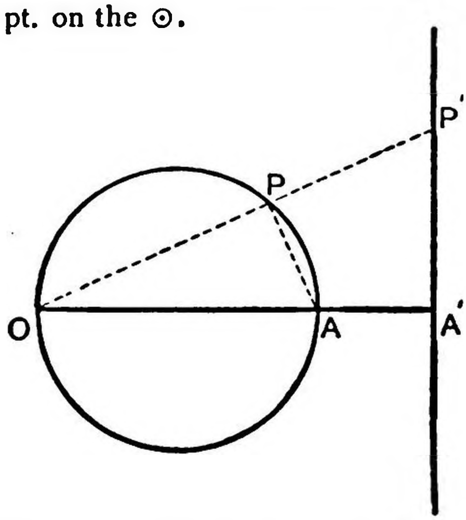
By the converses of III. 35 and III. 36 any two points P, Q, and their inverses P′, Q′, are concyclic.

N.B.—In the short discussion which follows on inversion we shall confine our demonstrations to the case in which OP′ is measured along OP, and leave the cases in which it is measured in the opposite direction as exercises to the student.

PROPOSITION.

The inverse of a circle with respect to a point on its circumference is a straight line perpendicular to the diameter through the point.

Let A be the other end of the diamr. through the pole of inversion O and P be any other pt. on the ⊙.



Along OA, OP, produced if necessary, take OA′, OP′ such that

$$OP.OP' = OA.OA',$$
$$= \text{constant of inversion}.$$

Join AP, A′P′. ∵ OP, OP′ = OA, OA′,

∴ APP′A′ is a cyclic quadl.,

∴ ∠OA′P′ = ∠OPA;

∴ A′P′ is ⊥r to OA′.

But A′ is a fixed pt. on OA or OA produced,

∴ P′ lies on a fixed st. line.

Ex. 479.—**Prove the same proposition when OP′ is measured in the opposite direction to OP.**

PROPOSITION.

The inverse of a given straight line with respect to an external point is a circle through that point whose diameter through the point is perpendicular to the given straight line.

Let A′P′ be the given st. line, A′ the projn. on it of the pole of inversion O, P′ any other pt. on it; along OA′, OP′, produced if necessary, take OA, OP such that

$$OP'.OP = OA'.OA = \text{constant of inversion.}$$

Join AP. ∵ OP.OP′ = OA.OA′,

∴ APP′A′ is a cyclic quadl.,

∴ ∠OPA = ∠OA′P′,

which is a rt. ∠.

∴ P lies on a ⊙ whose diamr. is OA.

Ex. 480.—**Prove the same proposition when OP is measured in the opposite direction to OP′.**

Ex. 481.—**Any circle and straight line being given, a pole and constant of inversion can be found such that each is the inverse of the other.**

The pole is always one end of that diameter of the circle perpendicular to the line.

If the straight line is a tangent to the circle, the pole is the end of the diameter opposite to the point of contact, the diameter itself being the radius of inversion.

If the straight line cuts the circle, either end of the diameter may be taken as the pole, the radius of inversion being the line joining the pole to either point of section.

Ex. 482.—In the triangle ABC, BQ and CR are the perpendiculars on CA, AB. If we take A as pole, and the tangent from A to the circle, through B, C, Q, R as radius of inversion, show that

(1) BC inverts into the ⊙ through A, Q, R,
(2) CR „ „ A, B, Q,
(3) BQ „ „ A, R, C,
(4) QR „ „ A, B, C,

and hence deduce the existence of the ortho-centre.

The ⊙s through A, B, Q, *and* A, C, R, *intersect in* P *the projn. of* A *on* BC. *Hence the inverse* T *of* P *must lie both on* BQ *and* CR.

$\therefore$ AP *passes through the cross of* BQ, CR.

Ex. 483.—The circum-circle of an isosceles triangle can be inverted into the indefinite straight line formed by producing the base each way.

Ex. 484.—A given circle is inverted into a straight line, the point P′ on the line being the inverse of the point P on the circle. Show that another circle can be described to touch the given circle at P and the straight line at P′.

Show also that the same constant and pole of inversion being chosen, the new circle inverts into itself.

Ex. 485.—A given straight line is inverted into a circle, the point P′ on the circle being the inverse of the point P′ on the straight line. Show that another circle can be described touching the given straight line at P and the circle at P′.

Show also that the same constant and pole of inversion being chosen, the new circle inverts into itself.

PROPOSITION.

The inverse of a circle with respect to an external or internal point is another circle.

Let O be a point external to the ⊙ APB. Let AB be the ends of the diamr. which, when produced, passes through O.

Take A′, B′, P′, inverse pts. of A, B, P.

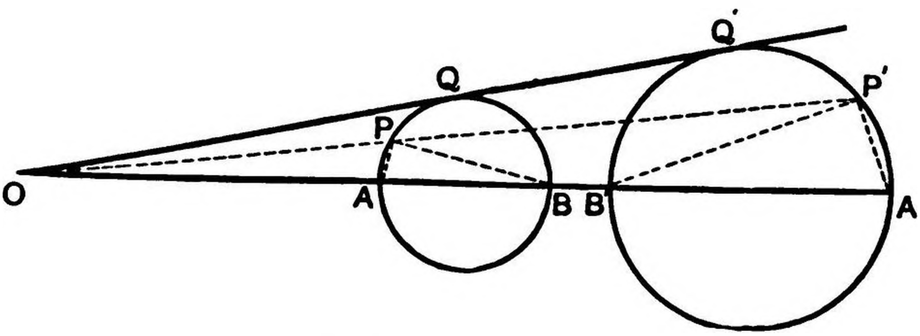
Then $\because$ OP.OP′ = OA.OA′,

$\therefore$ A, P, P′, A′ are concyclic

$\angle$OA′P′ = $\angle$OPA.

Simy., $\angle$A′B′P′ = $\angle$BPP′.

$\therefore$ ∠s OA′P′, A′B′P′ = ∠s OPA, BPP′
= a rt. ∠ ($\because$ APB is a rt. ∠),



$\therefore$ ∠ A′P′B′ is a rt. ∠;
$\therefore$ locus of P′ is a ⊙ on A′B′ as diamr.

The above demonstration and diagrams might require some slight modifications if the value of the constant of inversion or the position of the pole were changed.

Note that if P be on the part of circle APB which is *convex* to O, P′ will be on the part of circle A′P′B′ which is *concave* to O, and *vice versâ.*

Ex. 486.—If OQ be a tangent from O to circle APB, and Q′ the inverse of Q, then OQ′ is a tangent to circle A′P′B′.

Prove (1) *with,* (2) *without, the use of the 'Method of Limits.'*

Ex. 487.—Two circles being given, a pole and constant of inversion can be found such that each is the inverse of the other.

The pole will in general be the cross of either the external or the internal common tangents.

Ex. 488.—If p, p′ are the other pts. of section of OPP′ with the ⊙s APB, A′P′B′ respectively (diagram above),

Show that (1) AP ∥ B′p′
Ap ∥ B′p′
BP ∥ A′p′
Bp ∥ A′p′
(2) tangts. at P, p are ∥ respectively to tangents p′, P′.
(3) segt. PQp is similar to segt. P′Q′p′.

(∠OPA = ∠OA′P′ $\because$ A, A′, P′P *are concyclic,*
= ∠Op′B′ $\because$ B′, A′P, ′p′ *are concyclic.*

Note that AP, B′P′ *are both* antiparallel *to* A′P′ *with respect to* O. *See* p. 160, *and* Exx. 329, 362.)

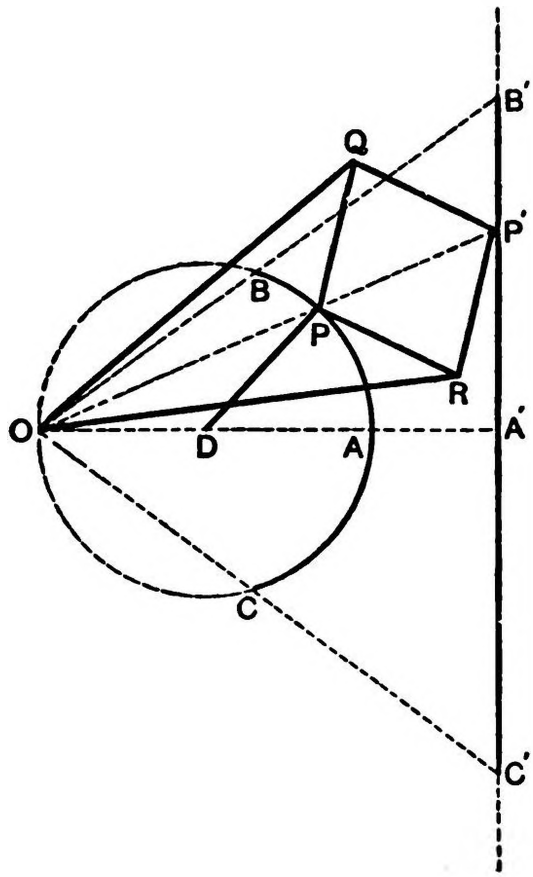
Ex. 489.—A given circle is inverted into another circle, the point P′ on the second being the inverse of the point P on the first. Show that another circle can be described to touch the given circle at P and its inverse at P′.

Show also that the same constant and pole of inversion being chosen the third circle inverts into itself.

Ex. 490.—The Simson line of a point O with respect to a triangle ABC and the circum-circle are both inverted with respect to O. Show that the inverse of the circum-circle is a Simson line of O with respect to a triangle inscribed in the inverse of the given Simson line.

PEAUCELLIER'S CELL.

This consists of a framework of six rods, OQ, OR, PQ, PR, P′Q, P′R hinged at their extremities, and such PQ = PR = P′Q = P′R, and OQ = OR.



This arrangement is called a **linkage**, the rods being linked or hinged together.

(1) Since O, P, P′ are equidistant from Q and R they lie on the ⊥r bisector of QR.

∴ O, P, P′ are in a straight line, however the rods may move.

(2) Also rect. OP, OP′ = difference of sqs. on OQ, PQ, which is constant. (See Ex. 195.)

∴ if O be fixed and P be made to move along any curve, P′ will move along the inverse of that curve with respect to O.

If, therefore, we introduce another link DP fixed at a point D, and such that DP = DO, the point P will move along a circle which passes through O, and the point P′ will therefore describe a straight line perpendicular to OD. (See p. 251.)

Ex. 491.—A framework is formed of four rods, CD, DE, EF, FC in one plane hinged at their extremities, CD being equal to EF and DE equal to CF; CD, EF being opposite sides, and CF, ED diagonals of an axe.

Show that the mid-points O, P, P′, O′ of CD, CF, DE, EF are in a straight line, and that the rectangle OP, OP′ is constant.

The above linkage is called **Hart's Contraparallelogram.**

If Q and R are the mid-points of DF, CE it can easily be shown (Ex. 65) that

$$PQ = PR = P'Q = P'R$$

and that $$OQ = OR,$$

so that the relative position of the five points O, Q, R, P, P is the same as in Peaucellier's cell.

For further information on 'Linkages' the student is referred to an interesting little treatise by Mr. A. B. Kempe, *How to Draw a Straight Line.*

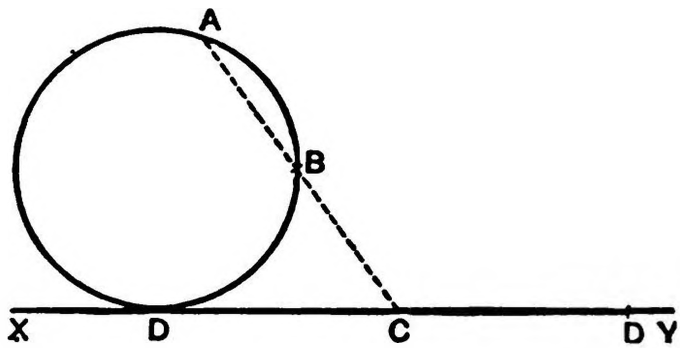
CONTACT PROBLEMS.

I. To describe a circle passing through **two given points** to touch

(*a*) a given straight line,

(*b*) a given circle.

(*a*) Let A and B be the two given pts., XY the given st. line, and suppose that AB is not ∥ to XY; let them meet in C.



Along XY take CD, CD′ such that sq. on CD = rect. AC, CB = sq. on CD′ and describe a ⊙ through A, B, D or A, B, D′.

It will touch XY by III. 37.

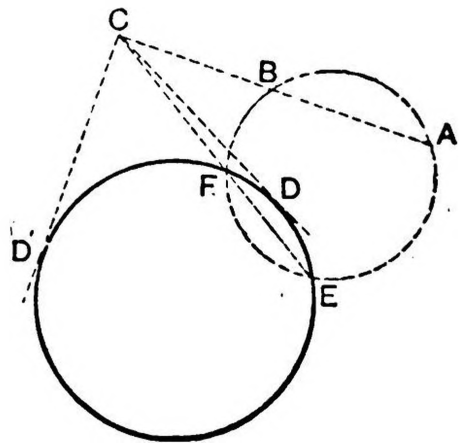
The case in which AB is parallel to XY is left as an exercise to the student.

Ex. 492.—**Find a point in a given straight line XY of unlimited length at which a given finite straight line AB subtends the greatest angle.**

If AB *meets* XY *in* C, XY *is divided into two parts,* CX, CY, *in each of which there exists a point at which the angle subtended by* AB *is a maximum for that part.*

Ex. 493.—D is any point in the side AB of a triangle ABC; show how to draw a straight line through D, cutting AC in E, so that DE may be equal to the sum of the perpendiculars let fall from D and E upon BC.

(*b*) Let A and B be the two given pts., DEF the given ⊙. Take any pt. E on ⊙ DEF and describe a ⊙ through A, B, E. If this does not touch ⊙ DEF let it cut it in F; and suppose that EF is not ‖ to AB; let them meet in C.



From C draw the tangents CD, CD′ to D, E, F, and describe a ⊙ through A, B, D or A, B, D′.

It will touch ⊙ DEF.

Sq. on CD = rect. CE.CF [III. 36.

= rect. CB.CA; [III. 36, Cor.

∴ CD touches ⊙ through A, B, D.

Similarly CD′ touches ⊙ through A, B, D′.

The case in which EF is parallel to AB is left as an exercise to the student.

Note that one of the tangents, CD′, from C might be in a straight line with AB, in which case it would be impossible to describe a circle through A, B, D′.

Ex. 494.—**Find the points on a given circle (DEF) at which a given finite straight line (AB) subtends the greatest and least angles.**

If AB when produced cuts circle DEF it will divide it into two parts, in each of which there exists a point at which the angle subtended by AB is a maximum for that part.

If AB when produced does not meet circle DEF, there exist two points, at one of which the angle subtended by AB is a maximum, at the other a minimum. (Compare Ex. 492.)

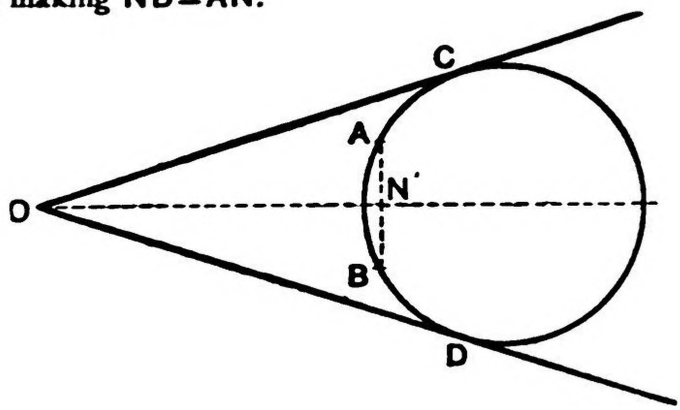
II. To describe a circle to pass through a given point and to touch,

(*c*) **two given straight lines,**

(*d*) **a given straight line and a given circle,**

(*e*) **two given circles.**

(*c*) Let A be the given pt., OC, OD the two given st. lines. Draw AN ⊥r to the intl. bisector of the ∠COD in which A lies, and produce it to B, making NB=AN.



Describe a ⊙ to pass through A and B, and touch OC at C:

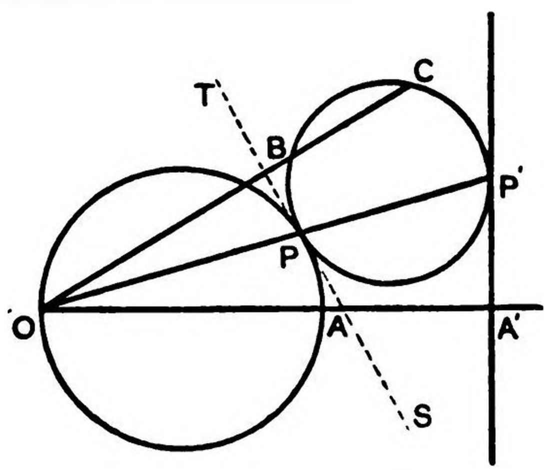
(*a*) it shall also touch OD.

For its centre E lies on the line ON. And if we draw ED ⊥r to OD, it =EC,

and ∴ the ⊙ touches OD at D.

The problem is thus reduced to (*a*).

There will therefore be two solutions.



(*d*) Let B be the given pt., A'P' the given st. line, and OAP the given ⊙. Let the diamr. OA of the ⊙ which is ⊥r to A'P' meet it in A'. Join OB and produce it to C, such that OB.OC=OA.OA'. [I. 45.

Describe a ⊙ through B, C touching A′P′ (*a*).

It shall also touch ⊙ OAP.

Let P′ be its pt. of contact with A′P′, and let OP′ cut ⊙ OAP at P. Join AP.

∵ ∠s at A′ and P are rt. ∠s,

∴ A, A′, P, P′ are concyclic;

∴ rect. OP. OP′ = rect. OA.OA′

= rect. OB.OC.

∴ ⊙ through B, C, P′ passes through P.

It will also touch circle OPA at P.

For let TPS be the tangent at P to ⊙ OPA

∠SP P′ = ∠TPO

= ∠OAP in alt. seg.

= int. opp. ∠OP′A′ of cyclic quadl. AA′PP′,

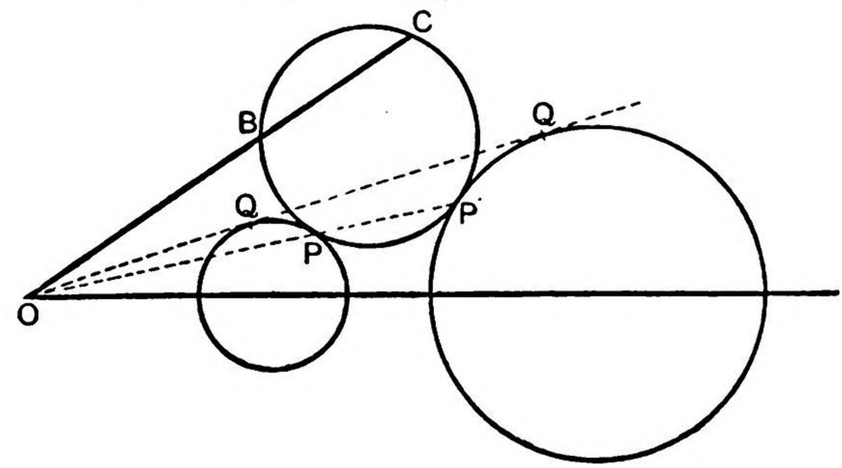
∴ TPS also touches ⊙ BCP′P.

The problem is thus reduced to (*a*).

There are thus two solutions, taking O at one end of the diameter perpendicular from the given straight line.

There will in general be two other solutions obtained by taking O at the other end.

Ex. 495.—Obtain two other solutions of this problem, distinguishing the cases in which the given straight line (1) does not meet, (2) cuts, (3) touches, the given circle.



(*e*) Let B be the given pt.; Q, Q′ the pts. of contact of a common tangent OQQ′ to the two given ⊙s; O the pt. where this common tangent cuts the st. line through the centres.

Join OB, and along it take OC such that OB.OC = OQ.OQ′.

Describe a ⊙ through B, C to touch one of the given ⊙s at P′.

Then ⊙ BCP′ shall also touch the other given ⊙.

For OP′ will meet the other given ⊙ at a pt. P such that OP.OP′

=OQ.OQ′,

=OB.OC.

∴ ⊙ BCP′ passes through P.

Also it touches the other given ⊙ at P.

For if it met it at any other pt. R besides P, the pt. R′ on OR such that OR.OR′=OP.OP′ would lie on ⊙BCP′ and on the first given ⊙, which is impossible, since they touch at P′.

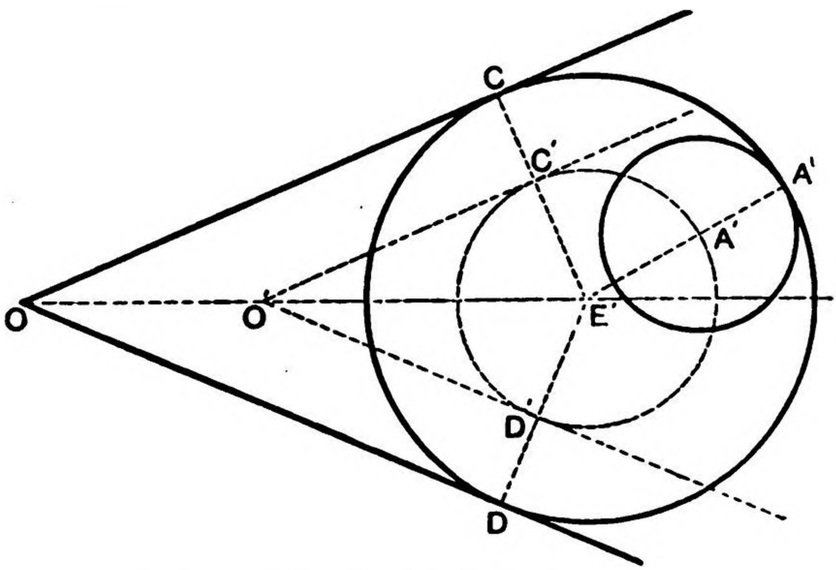
III. To describe a circle to touch

(*f*) **two given straight lines and a given circle,**

(*g*) **two given circles and a given straight line,**

(*h*) **three given circles.**

(*f*) Let OC, OD be the two given st. lines, A′ the centre of the given ⊙.



Draw two st. lines O′C′, O′D′ ‖ to OC, OD at a distance from them equal to the radius A′A of the given ⊙, and either on the same side of OC, OD as A′ or on the opposite side.

Describe a ⊙ through A′ touching O′C′, O′D′ at C′, D′. Let E be its centre.

Produce EC′, ED′ to cut OC, OD in C and D, and produce EA′ to cut the given ⊙ in A.

Then EC′=EA′=ED′,

and C′C=A′A=D′D;

∴ EC=EA=ED

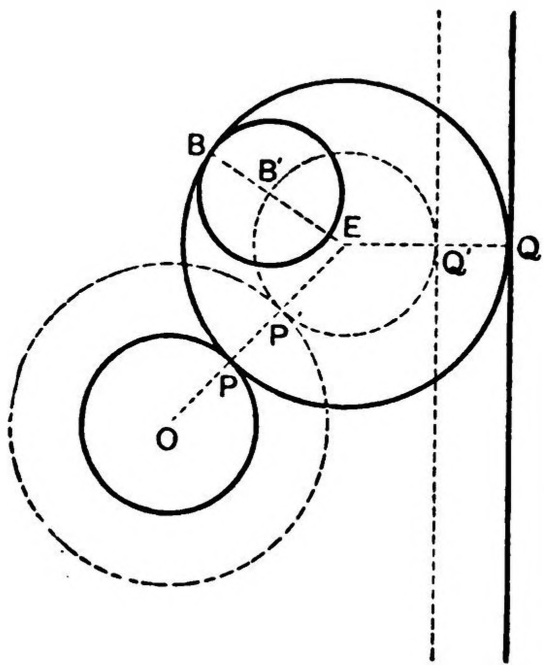
∴ a ⊙ can be described to touch the lines and given ⊙ as reqd. with centre E and radius EC, EA, or ED.

(*g*) Let the two ⊙s be unequal, and let O and B′ be the centres of the larger and smaller ⊙s respectively.

With centre O and radius equal to the sum of the two given radii, describe a ⊙, and on the same side of the given line as O draw a ∥ to the given st. line at a distance from it equal to the radius of the smaller ⊙.

Describe a ⊙ to pass through B′ and touch the new ⊙ and new st. line at P′ and Q′ by (*d*).

Let E be its centre, EQ′Q perpr. on the given st. line at Q, and the ∥ at Q′. Join EO, cutting the larger given circle at P. Join EB′ and produce it to cut the smaller given ⊙ at B.



A ⊙ can be described with centre E touching the given ⊙s at B and P, and the given st. line at Q.

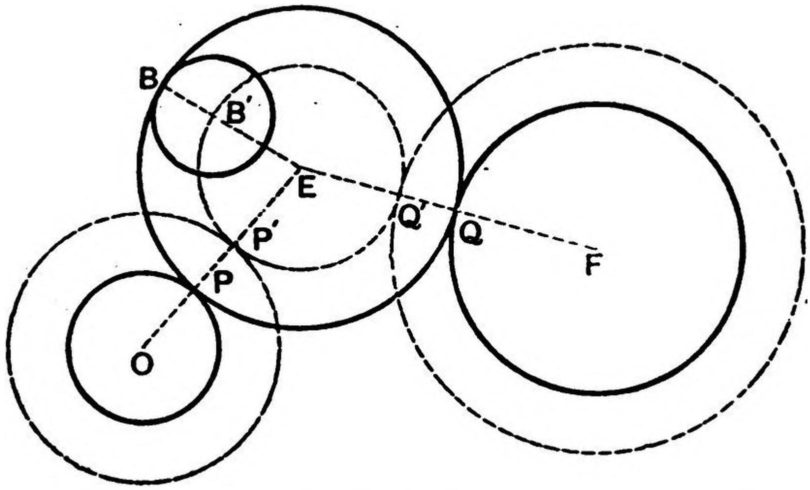
The proof resembles that of (*f*), and is left as an exercise to the student.

(*h*) Let the three ⊙s be unequal; B′ the centre of the smallest; O and F the centres of the other two; B, P, and Q points on their ○ces.

With centre O and radius equal to the sum of OP, BB′, describe a ⊙.

With centre F and radius equal to the sum of FQ, BB′, describe another ⊙.

Describe a ⊙ to pass through B′ and touch the two new ⊙s in P′, Q′: let E be its centre.



Produce EB′ to cut the smallest given ⊙ in B, and let EO, EF cut the other two given ⊙s in P and Q.

A ⊙ can be described with centre E touching the three given ⊙s in B, P, Q.

The proof resembles that of (*f*), and is left as an exercise to the student.

The number of solutions, the nature of the contact, and the peculiarities of construction necessitated by the various relative positions of the given lines, have been in several cases left to the student to examine for himself.

Contact with two given circles is best treated after some study of proportion.

With respect to the Solutions to (*f*), (*g*), (*h*), the attention of the student is drawn to the following passage from Petersen's *Methods and Theories for the Solution of Problems of Geometrical Constructions* :—

'A certain method, which may be classed as *parallel translation*, is often applied to problems where circles shall touch other circles or straight lines: it consists in diminishing *the radius of a circle to naught, the circle becoming a point, while simultaneously the other lines and the other circles follow; the former without changing their direction, the latter without changing their centre.* Having thus substituted a point for a circle, the problem is generally reduced to a simpler one, the other given conditions remaining unaltered.'

Ex. 496.—**To describe a circle which shall touch a given circle and touch a given straight line at a given point.**

Reduce the problem by Petersen's 'parallel translation' to the following simpler one :—

To describe a circle which shall *pass through a given point* and touch a given straight line at a given point.

Ex. 497.—To describe a circle which shall touch two given circles, one of them at a given point.

This problem can also be reduced to a simpler one by 'parallel translation.'

The student may be interested in the following passage from Montucla's account of Apollonius :—

'A quarrel which Vieta had with Adrian Romanus, a skilful Geometer of the Netherlands, gave him the opportunity of proposing the principal, in fact the only difficult problem in this book (Vieta's *Apollonius Gallus*); *i.e. Three circles being given, to find a fourth which touches them all.* It was but a poor solution of it which Romanus gave, determining the centre, as it is obviously possible to do by the intersection of two hyperbolas. For the problem is a plane one, and it ought therefore to be solved by Elementary Geometry. Vieta succeeded in effecting such a solution very elegantly. His solution is the same as that given in Newton's *Arithmetica Universalis* (Prob. 47). Another is given in the first book of the *Principia* (Lemma 16), where this question is required for some determinations in Physical Astronomy. Newton here reduces with remarkable dexterity the two solid loci of Romanus to the intersection of two straight lines. Descartes gave some attention to this problem, which does not yield readily to algebraical analysis, and of the two solutions which he found he acknowledges that one gave rise to such a complicated expression that he would not undertake to construct it in a month. The other, though less involved, is sufficiently so to deter Descartes from using it. . . .

'The Princess Elizabeth of Bohemia, who, as is well known, honoured our philosopher with her correspondence, deigned to communicate a solution, but as it is derived from the algebraical calculus, it labours under the same disadvantages as that of Descartes.'

(*Histoire des Mathématiques*, Part I. Liv. iv. § 7.)

MISCELLANEOUS EXERCISES.—VI.

(Books I. II. III.)

In the following Exercises the word 'cross' is frequently used, after a suggestion of Mr. R. C. J. Nixon, for 'intersection.'

Ex. 498.—The projections of the vertex A of the triangle ABC on the external and internal bisectors of the angles B and C lie on the straight line through the mid-points of the sides AB, AC. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

Ex. 499.—If two opposite sides of a quadrilateral are equal they are equally inclined to the straight line through the mid-points of the other two sides. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

Ex. 500.—If a quadrilateral be formed by drawing the internal bisectors of the angles of a given quadrilateral, its opposite angles are supplementary.

When the given quadrilateral is a parallelogram the other is a rectangle whose diagonals are parallel to the sides of the parallelogram.

When the given quadrilateral is a rectangle the other is a square. (Rouché et de Comberousse, *Traité de Géométrie Élémentaire.*)

Ex. 501.—Show that a convex polygon of an odd number of sides can be constructed if we are given the mid-points of its sides. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

Ex. 502.—In a right-angled triangle the difference of the squares on the segments into which the hypotenuse is divided by the projection on it of the mid-point of one of the sides containing the right angle is equal to the square on the other side. (Rouché et de Comberousse, *Traité de Géométrie Élémentaire.*)

Ex. 503.—When the vertices B and C of a triangle ABC remain fixed and the vertex A describes a given straight line BX, the centroid and mid-centre[1] of ABC also describe straight lines. (Professor Neuberg, *Educational Times.*)

Ex. 504.—Find a point D in AB produced such that the rectangle AD, DB shall be equal to the square on a given straight line greater than half AB.

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[1] See p. 235.

Ex. 505.—If a straight line be divided into any two parts, the rectangle contained by the diagonals of the squares on the whole line and one of the parts is equal to twice the rectangle contained by the two parts.

Ex. 506.—Show how to inscribe the least possible rhombus in a given rhombus.

Ex. 507.—From the vertices of a triangle ABC are drawn AE perpendicular to AC, meeting CB in E, BD perpendicular to AC, and CF perpendicular to AB, any of the lines being produced if necessary. Show that rectangle AE.CD = rectangle AB.CF.

Ex. 508.—In a given straight line AB any point C is taken, and equilateral triangles ADC, CEB are constructed on the same side of AB. Show that the locus of the mid-point of DE is a straight line parallel to AB.

Show also that the centre of the circum-circle of triangle CDE is a fixed point.

Ex. 509.—**The sum of the squares on the four sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the join of the mid-points of the diagonals.**

Ex. 510.—If the sum of the squares on the sides of a quadrilateral be equal to the sum of the squares on its diagonals the quadrilateral is a parallelogram.

Ex. 511.—**A point moves so that the sum of the squares of its distances from four given points is constant. Show that its locus is a circle.**

Ex. 512.—**The sum of the squares of the distances of a point from the vertices of a given triangle is a minimum when the point is the centroid of the triangle.**

Ex. 513.—Two circles touch each other externally at C. Show how to draw a straight line through C to meet the circumferences at A and B respectively, so that AB may equal a given straight line.

Ex. 514.—Construct a triangle having given its perimeter and its angles.

Ex. 515.—Construct a triangle having given its three ex-centres.

Ex. 516.—A and B are two given points, XY a given indefinite straight line. Find a point C on XY such that angle ACX equal twice angle BCY. (Vuibert's *Questions de Mathématiques Élémentaires.*)

Ex. 517.—From any point in the circumference of the larger of two concentric circles tangents are drawn to the inner. A third tangent is

drawn to that part of the inner circumference which is convex to the aforesaid point. Show that the perimeter of the triangle thus formed is invariable.

Ex. 518.—P is a point outside a given circle. Show that two straight lines and two only can be drawn though P such that the parts of them intercepted by the circle are equal.

Ex. 519.—O is the mid-point of AB, the common hypotenuse of two right-angled triangles ACB, ADB; from C and D straight lines are drawn perpendicular to OC, OD to intersect at P. Show that PC = PD.

Ex. 520.—D is a point on the arc BC of the circum-circle of an equilateral triangle ABC. BE is drawn parallel to CD to meet AD in E. Show that the locus of E is an arc equal to BC.

Ex. 521.—D is a point on the arc BC of the circum-circle of an equilateral triangle ABC. Show that DA = DB + DC.

Ex. 522.—A triangle ABC has each angle less than the exterior angle of an equilateral triangle. Equilateral triangles BDC, CEA, AFB are described on BC, CA, AB on the sides remote from A, B, C respectively. Show that AD, BE, CF are concurrent.

Ex. 523.—If O be the point at which AD, BE, CF in the last exercise intersect, show that

$$AD = BE = CF = OA + OB + OC.$$

Ex. 524.—A, B, C, D are four concyclic points. Show that the centroids of triangles BCD, CDA, DAB, ABC are also concyclic. (Professor Bordage, *Educational Times.*)

Ex. 525.—If two segments of circles have a common chord AB, and any points P and Q are taken on their arcs, the locus of the cross of the bisectors of the angles PAQ, PBQ is another arc of a circle having the same chord AB.

Ex. 526.—Three circles BDC, CEA, AFB intersect at the same point O, and AO, BO, CO are produced to cut the circumferences in D, E, F. Show that the triangles BDC, CEA, AFB are equiangular to one another.

Ex. 527.—If on the sides of triangle ABC there be described any three equiangular triangles BDC, CEA, AFB, either all externally or all internally, having their angles in the same order of rotation, and the angles contiguous to the same corner of ABC equal to each other, prove that AD, BE, CF meet in a point O which is also a common point of intersection of the circles BDC, CEA, AFB. (Morgan Jenkins, *Educational Times.*)

For a special case see Ex. 522.

Ex. 528.—The sides BC, CA, AB of triangle ABC are cut by a straight line in L, M, N, and lines through A, B, C parallel to LMN cut the circum-circle of triangle ABC in P, Q, R. Show that PL, QM, RN all cut the circle ABC in the same point. (J. L. M'Kenzie, *Educational Times.*)

Ex. 529.—From any point P in the bisector of the angle A in a triangle ABC, perpendiculars PA′, PB′, PC′ are drawn to BC, CA, AB. Show that PA′, B′C′ intersect on the median through A. (E. Rutter, *Educational Times.*)

Through the cross Q *of* PA′, B′C′ *draw a parallel* bc *to the base, then* PbC′Q, PB′cQ *are cyclic quadrilaterals, and it can be shown that* bQ, cQ *subtend supplementary angles in equal circles.*

Ex. 530.—ACP is a triangle having the angle ACP bisected by a straight line which meets AP in D. Describe a circle about ACD, and in its circumference take points B and E such that A, B, C, D, E are in order. Show that the angle at P equals the difference of the angles in the segments ABC, DEA.

Ex. 531.—The bisectors of the angles made by the opposite sides of a cyclic quadrilateral are at right angles to each other. (Catalan's *Théorèmes and Problèmes de Géométrie Élémentaire.*)

Ex. 532.—If the diagonals of a cyclic quadrilateral are at right angles to each other, the straight line through the cross of the diagonals perpendicular to one of the sides bisects the side opposite. (Brahmegupta's Theorem.)

Ex. 533.—If A, B, C, D are concyclic, the ortho-centres of the triangles ABC, CDA, DAB, ABC are the vertices of a quadrilateral congruent with ABCD. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

See Exx. 424-426.

Ex. 534.—If the diagonals AC, BD of a cyclic quadrilateral cut at right angles:—

(1) The projections on the sides of the cross of the diagonals are the vertices of a quadrilateral which can have one circle described about it and another inscribed in it.

(2) These four projections and the mid-points of the sides of the given quadrilateral are concyclic.

(3) The circle on which the eight points in (2) lie is the same for all cyclic quadrilaterals having the same circum-circle as the one given whose diagonals intersect at right angles at the same point. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

Ex. 535.—In a given circle draw a triangle one of whose sides passes through a given point, and whose other two sides are parallel to two given straight lines.

Find under what circumstances the two sides will contain an angle equal (1) to the acute angle, (2) to the obtuse angle, between the given lines.

Ex. 536.—Construct a triangle having given the base, the vertical angle, and the length of the line from the vertex to the base bisecting the vertical angle.

Ex. 537.—CA and CB are tangents to a circle whose centre is O ; AOD and EOF are diameters perpendicular to each other. Draw CP parallel to AD to meet EF or EF produced in P. Show that PB is parallel to OC, and that P, B, D are collinear.

Ex. 538.—If A, B, C, D are concyclic, the in-centres of the triangles BCD, CDA, DAB, ABC are the vertices of a rectangle. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

Ex. 539.—AB is a fixed chord in a circle APQB ; PQ another chord of given length. Show that if AP, BQ meet in R, R will lie on a fixed circle whatever be the position of PQ.

Ex. 540.—EF is the common chord of two circles EACF, EDBF: through E are drawn the straight lines AEB, CED, such that C, F, B lie in a straight line perpendicular to EF. Show that CA, FE, BD, when produced, all pass through the same point.

Ex. 541.—Any circle is drawn through the vertex of a given angle. Find the locus of the ends of that diameter of the circle which is parallel to the join of the points where it cuts the arms of the angle. (Professor Mannheim, *Educational Times.*)

The locus consists of a pair of straight lines at right angles to each other.

Ex. 542.—P is any point within the angle A formed by two straight lines AB, AC, to which PB, PC are perpendicular. Any point Q is taken such that angle QAB = angle PAC. Show that the line through the mid-point of PQ perpendicular to BC bisects BC. (Professor Hudson, *Educational Times.*)

First take Q *on circum-circle of cyclic quadrilateral* PBAC.

Ex. 543.—AB is a diameter and C any point on the circumference of a circle ABC. With C as a centre a circle is described to touch AB. Show that the other tangents to the latter circle from A and B are parallel.

Ex. 544.—Given two opposite sides, the sum of the other two, and the circum-circle of a cyclic quadrilateral, construct the quadrilateral.

To what extent is the solution ambiguous?

Ex. 545.—ABC is *any* right-angled triangle on a given hypotenuse AB. Along AC, BC are taken AD, BE equal to BC, AC respectively. Find the loci of the points D and E, and show that DE passes through a fixed point. (Vuibert's *Questions de Mathématiques Élémentaires.*)

Ex. 546.—MN is a diameter of a circle; P an external point; PB perpendicular to MN; PHK cuts the circle. Show that square on PB equals sum or difference of rectangle PH, PK and rectangle MB, BN.

Ex. 547.—ABCD is a quadrilateral inscribed in a circle, of which AB is a diameter: AD, BC meet in E. Show that rectangle AD, AE + rectangle BC, BE = square on AB.

Ex. 548.—A circle passes through A, the vertex of an equilateral triangle ABC, and meets AB and AC produced in D and E respectively; it also cuts off BF and CG in BC produced both ways. Show that the difference between BD and CE is equal to the difference between BF and CG.

Ex. 549.—The difference of the squares of two intersecting chords of a circle is equal to the difference of the squares on the differences of their segments. (Rouché et de Comberousse, *Traité de Géométrie Elémentaire.*)

Ex. 550.—Let CA be radius of a circle whose centre is C; draw a radius CB perpendicular to CA, and show how to draw a straight line APQ cutting the circle in P and CB in Q, such that PQ may be of given length.

Take DE *equal to the reqd. length of* PQ; *bisect it in* F, *and take* G *in* DE *produced, such that* $FG^2 = AB^2 + FE^2$. *Then* EG = AP.

Ex. 551.—A triangle ABC is turned about a fixed point X in its plane; if P, Q, R be the intersections of the sides BC, CA, CA respectively with $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, the sides of the triangles, in any other position, show that the angles of triangle PQR are invariable.

Find also what point X must be with reference to the triangle in order that it may be the (1) circum-centre, (2) in-centre, (3) ortho-centre, (4) centroid of PQR. (Professor de Longchamps, *Educational Times.*)

Ex. 552.—If a triangle ABC turns around its circum-centre O into the position A′B′C′, and if AB, A′B′ meet in a; BC, B′C′ in b; CA, C′A′ in c, prove that the triangle abc will have O for its ortho-centre. (N'Importe, *Educational Times.*)

Ex. 553.—If a cyclic quadrilateral ABCD turn about its circum-centre O into the position A′B′C′D′, and *a* is the cross of AB, A′B′; *b* that of BC, B′C′; *c* that of CD, C′D′; *d* that of DA, D′A′; *e* that of BD, B′D′;

and f that of AC, A′C′, show that $abcd$ is a parallelogram, and that the sides ab, ad are respectively perpendicular to Of, Oe. (Professor Ignacio Beyens, *Educational Times.*)

Ex. 554.—Through the mid-point of each side of a triangle are drawn perpendiculars to the other two sides. Show that the two triangles thus formed have the same symmedian point. (See Ex. 258.)

Ex. 555.—Show that the symmedian point of a triangle is the centroid of its projections on the sides.

Ex. 556.—D, E, F are the points of contact of the in-circle of triangle ABC with BC, CA, AB. Through A is drawn PAQ parallel to BC, meeting DE, DF produced in P, Q. Show that AP = AQ, and hence that AD, BE, CF intersect in a point.

(AD *is a symmedian of triangle* DEF; see pp. 160, 161. *Similarly,* BE *and* CF *are symmedians of triangle* DEF. *Hence* AD, BE, CF *intersect at the symmedian point of triangle* DEF.)

Ex. 557.—AP, BQ, CR are the perpendiculars from A, B, C to the sides BC, CA, AB. Show that the projections of P on CA, AB; Q on AB, BC; R on BC, CA, all lie on the same circle. (Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.*)

This circle is sometimes called Taylor's Circle.

Ex. 558.—If a point be the centroid of its projections on the sides of a given triangle, it must be the symmedian point of the triangle.

Ex. 559.—Find a point within a triangle such that the sum of the squares of its distances from the sides is a minimum.

The point must be the centroid of its projections on the sides, and therefore the 'symmedian point' of the given triangle.

Ex. 559 (*a*).—D, E, F are the points of contact of an ex. ⊙ of △ABC with BC, CA, AB show that AD, BE, CF meet in a point. (Compare Ex. 556.)

Ex. 559 (*b*).—In Fig. of I. 47 if FG, HK be produced to meet in M then FC, AL, BK pass through the ortho-centre of △MBC. (Grunert's *Archiv.*)

THE HARPUR EUCLID

BOOK IV.

Book IV. is entirely devoted to the solution of Problems.

Of these, four relate to triangles, four to squares, four to pentagons; and there are four others.

They are all cases of the inscription of figures in, or the circumscription of figures about, circles.

Def.—**A straight line is said to be placed in a circle when its extremities are on the circumference of the circle.**

PROPOSITION 1.—Problem.

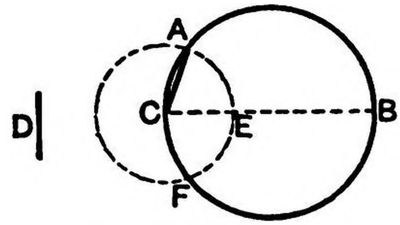
In a given circle to place a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given ⊙, D the given st. line; it is reqd. to place in ⊙ ABC a st. line equal to D.

Draw any diamr. BC.

If BC = D, the prob. is solved.

If not, from CB cut off CE equal to D.



With centre C and radius CE describe a ⊙ cutting ABC in A.

Join AC: AC shall be the line reqd.

For rad. CA = rad. CE

= D.

NOTE.

We solve a more definite problem than the one proposed: one end of the required chord may be *at any point on the given circle.*

Since the second circle cuts the given one in *two points*, we have shown that:—

If a straight line be less than the diameter of a given circle, two chords equal to it can be drawn from any point on the given circle. Compare III. 7 (2); III. 8 (3).

Ex. 560.—Construct a right-angled triangle, having given the hypotenuse and one side.

Ex. 561.—Through a given point draw a straight line, the part of which intercepted by a given circle shall be equal to a given straight line.

Def.—If the angular points of a rectilineal figure are on the circumference of a circle,

(1) the rectilineal figure is said to be inscribed in the circle;

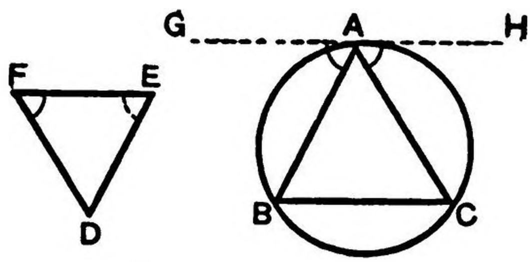
(2) the circle is said to be described about the rectilineal figure.

PROPOSITION 2. Problem.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given ⊙, DEF the given △: it is reqd. to inscribe in ⊙ ABC a △ equiangular to △ DEF.

Through A draw GAH, touching the ⊙. [III. 17.



At A make ∠ HAC equal to ∠ DEF,
and ∠ GAB equal to ∠ DFE.

Join BC; then ABC is the △ reqd.

For GAH is the tangent at A, and AC a chd. through A;

∴ ∠ HAC = ∠ ABC in alt. seg.; [III. 32.

but ∠ HAC = ∠ DEF; [Const.

∴ ∠ ABC = ∠ DEF.

Similarly $\angle$ ACB = $\angle$ DFE ;

$\therefore$ third $\angle$ BAC = third $\angle$ EDF ;

$\therefore$ $\triangle$ ABC, inscribed in $\odot$ ABC, is equiangular to $\triangle$ DEF.

Ex. 562.—Prove that any number of triangles equiangular to triangle DEF can be inscribed in circle ABC.

Ex. 563.—If another triangle equiangular to triangle DEF be inscribed in circle ABC, show that it is congruent with triangle ABC.

Ex. 564.—In a given circle inscribe an equilateral triangle.

Def.—If each side of a rectilineal figure touches a circle,

(1) the rectilineal figure is said to be described about the circle;

(2) the circle is said to be inscribed in the rectilineal figure.

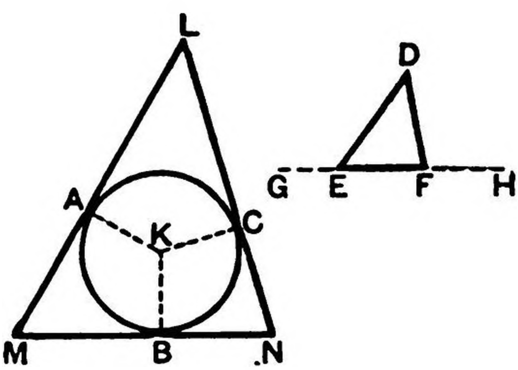
PROPOSITION 3. Problem.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given ⊙, DEF the given △: it is reqd. to describe about ⊙ ABC a △ equiangular to △ DEF.

Produce EF both ways to G, H.

Find the centre K of ⊙ ABC, and draw any radius KB.



Make ∠s BKA, BKC, on opposite sides of BK, equal to ∠s DEG, DFH.

At A, B, C draw tangents LM, MN, NL.

LMN is the △ reqd.

∵ LM is a tangent and KA the rad. to the pt. of contact A;

∴ the ∠s at A are right.

Similarly, the ∠s at B and C are right.

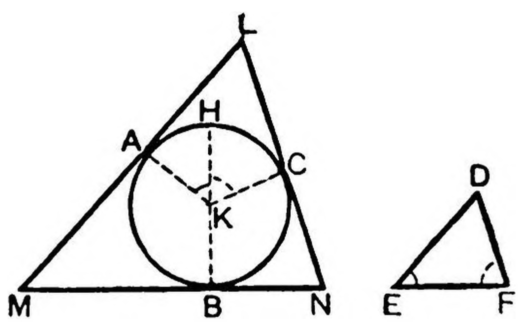
Now the four ∠s of quadl. AMBK = four rt. ∠s, [I. 32. Cor.
and two of these ∠s, KAM, KBM, are rt. ∠s ;
∴ ∠s AMB, AKB = 2 rt. ∠s,
= ∠s DEF, DEG.
But ∠AKB = ∠DEG ; [Const.
∴ ∠AMB = ∠DEF.
Similarly ∠CNB = ∠DFE ;
∴ third ∠L = third ∠D ; [I. 32.
∴ the △ LMN, described about ⊙ ABC, is equiangr. to △ DEF.

NOTE.

It is assumed that the tangents at A, B, C intersect so as to form a triangle LMN. The student should have no difficulty in proving this. Compare the construction given above with the following, taken from the *Text-Book of Geometry*, published by the Association for the Improvement of Geometrical Teaching.

Draw any diamr. BKH.

Make ∠s HKA, HKC on opposite sides of BH equal to ∠s E, F.



At A, B, C draw tangent forming the △ LMN.

This will be the △ reqd. ;
∵ ∠s at A and B are right ;
∴ AMBK is a cyclic quadl. ;
∴ ∠M = ext. ∠AKH,
= ∠E.
Similarly ∠N = ∠F ;
∴ third ∠L = third ∠D. [I. 32.

Ex. 565.—Show that any number of triangles equiangular to triangle DEF can be described about circle ABC.

Ex. 566.—If another triangle equiangular to triangle DEF be described about circle ABC, show that it is congruent with triangle LMN.

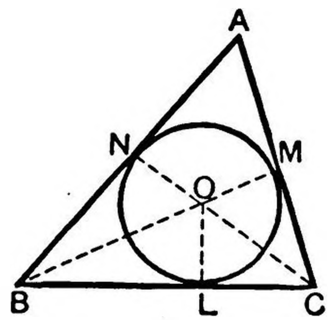
Ex. 567.—AK is produced to meet the circle in P: through P a tangent SPR is drawn, meeting MN, NL in S, R. Show that triangle RSN is equiangular to triangle DEF.

Ex. 568.—About a given circle describe a quadrilateral equiangular to a given quadrilateral.

PROPOSITION 4. PROBLEM.

To inscribe a circle in a given triangle.

Let ABC be the given △; it is reqd. to inscribe a ⊙ in it.
Bisect the ∠s ABC, BCA by the st. lines BO, CO.
From O draw OL, OM, ON ⊥r to BC, CA, AB:



In △s OLB, ONB.

∠OBL = ∠OBN, [Const.

rt. ∠OLB = rt. ∠ONB,

and OB (opp. to equal ∠s in each) is common;

∴ OL = ON.

Similarly OL = OM;

∴ the ⊙ described with centre O and radius OL will pass through L, M, N,

and ∵ the ∠s at L, M, N are right;

∴ it will touch the sides of △ ABC.

∴ this is the reqd. ⊙.

Ex. 569.—Show that the assumption that the bisectors of the angles ABC, BCA will meet is correct.

Ex. 570.—Show that OA bisects ∠CAB.

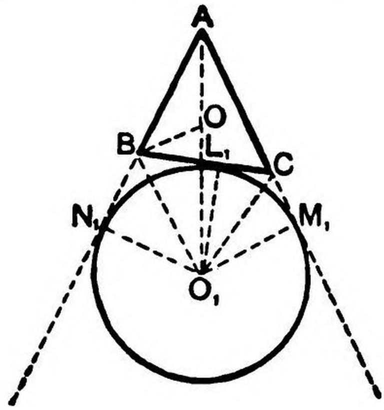
NOTE.

IV. 4 is a particular case of the general problem :—

To describe a circle touching three given straight lines which intersect each other, but not at the same point.

Produce AB, AC and bisect the exterior ∠s thus formed by BO_1, CO_1.

Drop ⊥rs O_1L_1, O_1M_1, O_1N_1 to BC, CA, AB.



By a demonstration similar to that of IV. 4 we can show that $O_1N_1 = O_1L_1 = O_1M_1$;

and hence that O_1 is the centre of a ⊙ which can be drawn touching BC at L_1, and CA, AB produced at M_1, N_1.

This circle is called an **escribed circle** of the triangle ABC.

There are evidently two other escribed circles of the triangle ABC.

We shall denote their centres by O_2 and O_3, and their points of contact by L_2, M_2, N_2, and L_3, M_3, N_3 respectively, M_2 being on CA, and N_3 on AB.

Note that, for the sake of brevity, LMN is often called the **in-circle**, and its centre O the **in-centre**, and its radius the **in-radius** of the triangle ABC; and that $L_1M_1N_1$, $L_2M_2N_2$, $L_3M_3N_3$ are called the **ex-circles**, their centres O_1 O_2 O_3 the **ex-centres**, and their radii the **ex-radii** of the triangle ABC.

Join AO_1. Then AO_1 can be shown to bisect the ∠BAC. Hence the centre O of the in-circle lies on AO_1 and can be found by bisecting ∠ABC by BO.

Hence A, O, O_1 are collinear.

Similarly B, O, O_2 are collinear,

and C, O, O_3 are collinear.

The student is recommended to draw a careful diagram for his own use, showing all these circles, their centres, and their points of contact.

Ex. 571.—The in-centre of an equilateral triangle is equidistant from the corners of the triangle.

Ex. 572.—If ABC is equilateral, $AO_1 = BO_2 = CO_3$. Also triangle LMN and triangle $O_1O_2O_3$ will be equilateral, and $OO_1 = OO_2 = OO_3$.

Ex. 573.—Draw a triangle equiangular to a given triangle, and having a given circle for one of its ex-circles.

Compare IV. 3. Show that there are three solutions.

Ex. 574.—Show how to describe a circle which shall cut off equal chords from the sides of a given triangle.

Ex. 575.—With three given points as centres, describe three circles to touch each other.

There are four solutions to this problem.

Ex. 576.—The diameter of the in-circle of a right-angled triangle and the hypotenuse are together equal to the sum of the other two sides.

Enunciate and prove a similar theorem for an ex-circle of a right-angled triangle.

In the following exercises the notation is that described in the notes to IV. 4.

Ex. 577.—Show that O is the orthocentre of $O_1\ O_2\ O_3$.

Ex. 578.—Show that $AM_1 = AN_1 = BL_2 = BN_2 = CM_3 = CL_3$ = semiperimeter of triangle ABC.

(This semiperimeter is usually denoted by s and the sides BC, CA, AB by a, b, c.)

Ex. 579.—Show that $AM = AN = BN_3 = BL_3 = CM_2 = CL_2 = s - a$.

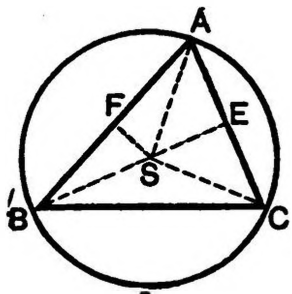
Ex. 580.—Find what lines in the figure are equal to $s - b$ and $s - c$.

Ex. 581.—Find the locus of O when the base and vertical angle of triangle ABO are given.

PROPOSITION 5. PROBLEM.

To describe a circle about a given triangle.

Let ABC be the given △; it is reqd. to describe a ⊙ about it.
Bisect CA, AB in E, F; at E and F erect ⊥rs ES, FS to CA, AB meeting in S. Join SA, SB, SC.



[These ⊥rs must meet, otherwise CA, AB, to which they are ⊥r, would be either ∥ or in the same st. line.]

In △s AFS, BFS

AF = BF, [CONST.

FS is common,

and rt. ∠ AFS = rt. ∠ BFS;

∴ SA = SB.

Similarly, SA = SC;

∴ the ⊙ described with centre S and rad. SA will pass through A, B, C, and will be the ⊙ reqd.

COR.—It is plain that

(1) if △ ABC is acute angled, S falls within it;
(2) if ∠ CAB is right, S falls on BC;
(3) if ∠ CAB is obtuse, S falls on the side of BC opposite to A; and conversely,
(4) if S falls within △ ABC, that △ is acute angled;
(5) if S falls on BC, ∠ CAB is right:
(6) if S falls on the side of BC opposite to A, ∠ CAB is obtuse.

An excellent example is here given of the Rule of Conversion. See pp. 185 and 217

If we show the truth either of the group (1), (2), (3), or the group (4), (5), (6), the truth of the other group follows immediately. See p. 185.

For the sake of brevity, the circle described about a triangle is often called the **circum-circle**, its centre the **circum-centre**, and its radius the **circum-radius** of the triangle.

IV. 5 is often set as follows :—

To describe a circle which shall pass through three given points not in the same straight line.

It includes, of course, the problem—

To find a point equidistant from three given points not in the same straight line.

N.B.—It follows from the demonstration of IV. 5 that the perpendicular bisectors of the sides of a triangle meet at a point, viz. the circum-centre of the triangle.

Pp. 99, 107 contain matter bearing directly on IV. 4, 5. If the student has not already thoroughly mastered their contents, it would be well for him to do so now.

Ex. 582.—The circum-centre of an equilateral triangle is also the in-centre.

Ex. 583.—The equilateral triangle described about a circle is four times that inscribed in the circle.

Ex. 584.—The square on the side of an equilateral triangle is three times the square on its circum-radius.

In the following Exercises the notation is that described in the notes to IV. 4.

Ex. 585.—AO is produced to cut the circum-circle of triangle ABC in S. Show that $SB = SO = SC$.

Ex. 586.—OO_1, OO_2, OO_3 are all bisected by the circum-circle of triangle ABC.

Use may be made of this theorem in Trigonometry.

Ex. 587.—The circum-centres of triangles O_1BO, O_2CA, O_3AB are on the circum-circle of triangle ABC.

PROPOSITION 6. PROBLEM.

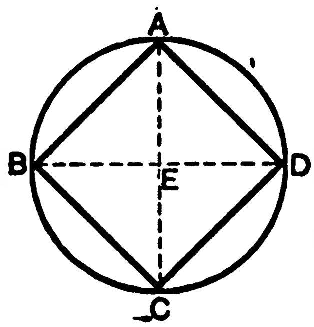
To inscribe a square in a given circle.

Let ABCD be the given ⊙; it is reqd. to describe a square in it.

Find the centre E of ⊙ ABCD.

Draw a diamr. AEC and the diamr. BED ⊥r to AEC.

Join AB, BC, CD, DA.



Then ABCD is the reqd. square.

For rt. ∠ AEB = rt. ∠ BEC = rt. ∠ CED = rt. ∠ DEA;

∴ arc AB = arc BC = arc CD = arc DA; [III. 26.

∴ chd. AB = chd. BC = chd. CD = chd. DA; [III. 29.

∴ ABCD is equilateral.

Again, each of the ∠s BCD, CDA, DAB, ABC being in a semi⊙, is a rt. ∠;

∴ ABCD is also rectangular;

and it has been inscribed in ⊙ ABCD.

Cor.—The circumference of a circle can be divided into 4, 8, 16, 32 . . . equal arcs.

Ex. 588.—All the squares that can be inscribed in the same circle are equal.

Ex. 589.—The square inscribed in a circle is double the square on the radius and half the square on the diameter.

Ex. 590.—If the ends of any two diameters of a circle be joined consecutively, a rectangle is inscribed in the circle.

Ex. 591.—Inscribe a rectangle in a given circle, with one side equal to a given straight line.

What limitation must be imposed on the straight line?

Ex. 592.—To inscribe a regular octagon in a given circle.

Ex. 592 (*a*).—If the Fig. of IV. 6 a straight line be drawn from A cutting the ⊙ABC in F and BD or BD produced in G, then rect. AF, AG = sq. ABCD.

(This special case of a more general theorem is given in Cardan's *De Proportionibus*, Lib. v. 192. Compare Exx. 479, 480, 481.)

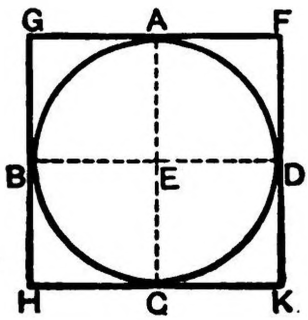
PROPOSITION 7. PROBLEM.

To describe a square about a given circle.

Let ABCD be the given ⊙; it is reqd. to describe a square about the ⊙ ABCD.

Find the centre E of ⊙ ABCD.

Draw any diamr. AEC and the diamr. BED ⊥r to AEC.



At A, B, C, D draw tangents forming the figure GHKF.

GHKF shall be the square required;

∵ GF touches the ⊙ at the end of diamr. AC;

∴ ∠s at A are rt.

Similarly, the ∠s at B, C, D are rt.

Again, ∵ ∠s AEB, EBG are rt. ∠s.

∴ GH is ∥ to AC.

Similarly, FK is ∥ to AC,

GF is ∥ to BD,

and HK is ∥ to BD.

Hence all the quadls. in the fig. are ∥gms.

Again, GF, HK each = diamr. BD, [I. 34.

and GH, FK each = diamr. AC;

∴ GHKF is equilateral.

Again, the ||gms. GE, AD, BC, EK have each an ∠ at E right;

∴ all their ∠s are rt.; [I. 46, Cor.

∴ GFKH is also rectangular;

∴ it is a square,

and it has been described about ⊙ ABCD.

Ex. 593.—Show that the assumption that the tangents at A and B meet is correct.

Ex. 594.—The square about a circle = twice the square in the circle = four times the square on the radius of the circle.

Ex. 595.—If a rectangle be described about a circle, it must be a square.

Note that any rectangle of any species can be inscribed in a circle, but only an equilateral one (*i.e.* a square) about a circle.

Ex. 596.—The regular octagon in a given circle is equal to the rectangle contained by the sides of the inscribed and circumscribed squares.

Ex. 597.—If tangents be drawn at the ends of any two diameters, a rhombus is circumscribed about the circle. Compare Ex. 590.

Ex. 598.—About a given circle to describe a rhombus with a diagonal equal to a given straight line.

Ex. 599.—To describe a regular octagon about a given circle.

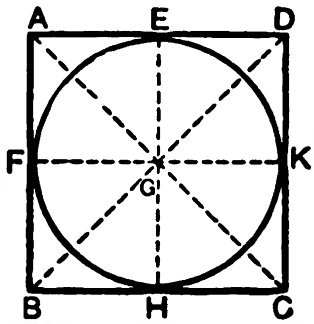
PROPOSITION 8. PROBLEM.

To inscribe a circle in a given square.

Let ABCD be the given square; it is reqd. to inscribe a ⊙ in it.

Bisect ∠s ABC, BCD by BG, CG.

Join GA, GD. Draw GE, GF, GH, GK ⊥r to DA, AB, BC, CD.



In △s ABG, BGC
AB = BC,
BG is common,
and ∠ABG = ∠GBC;
∴ ∠BAG = ∠BCG.

But ∠BCG is half of ∠BCD, which = ∠BAD;
∴ ∠BAG is half of ∠BAD;
∴ GA bisects ∠BAD.

Similarly GD bisects ∠CDA.

In △s FGB, BGH
∠FBG = ∠HBG, [CONST.
rt. ∠GFB = rt. ∠GHB,
and BG, opp. equal ∠s in each, is common;
∴ GF = GH.

Similarly GH = GK = GE.

The ⊙ described with centre G and radius GF will pass through E, F, H, K and touch the sides of ABCD,
∵ the ∠s at E, F, H, K are rt.;
∴ it is the ⊙ reqd.

NOTE.

We have adopted the above method of proof as it is the same as that employed by Euclid for IV. 13. The centre G of the in-circle of the square may also be found by joining opposite corners of the square. It is easy to show that the diagonals bisect the angles, and hence, by a demonstration like ours, that their cross G is equidistant from the sides.

Ex. 600.—If AD, AB be bisected in E, F, and through E, F, ||s be drawn to AB, AD, these ||s will intersect in G the centre of the in-circle.

Ex. 601.—Inscribe a circle in a given rhombus.

Ex. 602.—Inscribe a circle in a given kite.

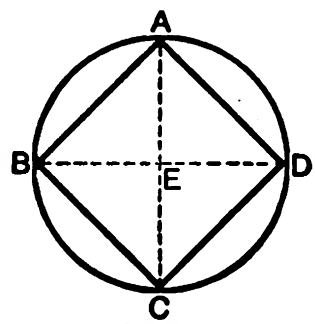
PROPOSITION 9. PROBLEM.

To describe a circle about a given square.

Let ABCD be the given square; it is reqd. to describe a ⊙ about it.

Bisect ∠s ABC, BCD by BE, CE.

Join EA, ED.



In △s ABE, CBE,
AB=CB,
BE is common,
and ∠ABE=∠CBE; [CONST.
∴ ∠BAE=∠BCE.
But ∠BCE is half of ∠BCD, which = ∠BAD;
∴ ∠BAE is half of ∠BAD;
∴ EA bisects ∠BAD.
Similarly ED bisects ∠CDA.
Again, ∠EBC=∠ECB (∵ each is half the ∠ of a sq.);
∴ EB=EC.
Similarly EC=ED=EA;
∴ the ⊙ described with centre E and radius EB will pass through A, B, C, D, and will ∴ be the ⊙ reqd.

NOTE.

We have adopted the above proof as it is the same as that used by Euclid for IV. 14.

Ex. 603.—The centre of the circum-⊙ of a square ABCD is the cross E of the diagls. AC, BD. (Fig. of IV. 9.)

Ex. 604.—The centre of the circum-⊙ of a square ABCD is the cross G of the lines EH, FK joining the mid-pts. of opposite sides. (Fig. of IV. 8.)

Ex. 605.—Show that a ⊙ cannot be described about a rhombus as defined by Euclid. (See pp. 97, 101.)

Ex. 606.—Show that if tangents be drawn at the corners of a square to its circum-⊙ another square will be formed whose area = twice that of the given square.

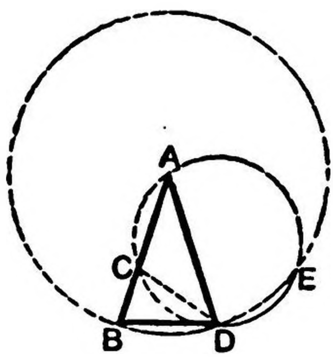
PROPOSITION 10. PROBLEM.

To describe an isosceles triangle having each of the angles at the base double the third angle.

Take any st. line AB; divide it at C so that rect. AB, BC= sq. on AC. [II. 11.

With centre A and rad. AB describe ⊙ BDE, in which place BD equal to AC. Join AD.

Then △ ABD is the reqd. △.



Join CD, and about △ ACD describe ⊙ ACD.

Rect. AB, BC=sq. on AC

=sq. on BD (∵ BD=AC);

∴ BD touches ⊙ ACD. [III. 37.

∴ ∠ BDC= ∠ CAD in alt. segt. cut off by DC.

But ∠ B is common to △s BCD, BAD;

∴ third ∠ BCD=third ∠ ADB [I. 32.

= ∠ ABD (∵ AD=AB);

∴ CD=BD [I. 6.

=CA; [CONST.

∴ ∠ CDA= ∠ CAD.

But ∠ BDC= ∠ CAD;

∴ whole ∠ BDA=twice ∠ CAD;

∴ also ∠ ABD=twice ∠ CAD.

Hence △ ABD is the one reqd.

NOTES.

The $\triangle$ BCD also possesses the required property, as is evident from the proof.

Since each angle at the base = twice $\angle$ A,
$\therefore$ the three $\angle$s of the $\triangle$ together = five times $\angle$ A,
$\therefore$ $\angle$ A = one-fifth of two rt. $\angle$s.

Ex. 607.—**Divide a right angle into five equal parts.**

Ex. 608.—Divide a rt. $\angle$ into fifteen equal parts.

Ex. 609.—On a given base describe an isosceles $\triangle$ having each of the $\angle$s at the base double of the vertical $\angle$.

Ex. 610.—Prove that $\angle$ ACD = three times each of the $\angle$s CAD, CDA.

Ex. 611.—On a given base describe an isosceles $\triangle$ having each of the angles at the base one-third of the vertical $\angle$.

Ex. 612.—Show that BD is the side of a regular decagon (ten-sided figure) inscribed in $\odot$ BDE.

Ex. 613.—Show that $\odot$ ACD cuts the $\odot$ BDE.

Ex. 614.—If E be the other pt. where the $\odot$s ACD, BDE cut, prove (1) that $\triangle$ AED is equiangular to $\triangle$ ABD; (2) that AE is $\parallel$ to CD.

Ex. 615.—If M be the mid-pt. of the minor arc AE, show that ACDEM is a regular pentagon.

N.B.—(1) **BD is the side of a regr. decagon inscribed in $\odot$ BDE.**
(2) BD „ „ **pentagon** „ $\odot$ ACD

Ex. 616.—Inscribe in $\odot$ ACD a $\triangle$ congruent with $\triangle$ ABD.

Ex. 617.—The circum-$\odot$s of $\triangle$s ABD, ACD are equal.

Def.—If a figure has all its sides equal to one another and all its angles equal to one another it is said to be 'regular.'

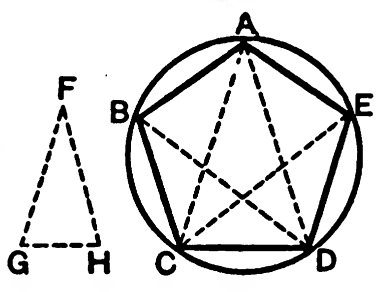
Thus an equilateral triangle is a 'regular triangle,' and a square is a 'regular quadrilateral.'

PROPOSITION 11. Problem.

In a given circle to inscribe a regular pentagon.

Let ABCDE be the given ⊙; it is reqd. to inscribe a regular pentagon in ⊙ ABCDE.

Describe an isosceles △ FGH having each of the ∠s G, H double the ∠ F.



Inscribe in ⊙ ABCDE a △ ACD such that
∠s CAD, ADC, DCA = ∠s F, H, G, [IV. 2.
so that each of ∠s ACD, ADC is double of ∠ CAD.
Bisect ∠s ACD, ADC by CE, DB.
Join AB, BC, CD, DE, EA.
Then ABCDE is the pentagon reqd.

∵ CE, DB bisect ∠s ACD, ADC, which each = twice CAD,
∴ ∠s ADB, BDC, CAD, ECD, ACE are all equal;
∴ arcs AB, BC, CD, DE, EA are all equal; [III. 26.
∴ chds. AB, BC, CD, DE, EA are all equal. [III. 29.
i.e. fig. ABCDE is equilateral.

Again each of the five ∠s EAB, ABC, BCD, CDE, DEA stands on an arc which is made up of three of the five equal arcs AB, BC, CD, DE, EA;

∴ ∠EAB = ∠ABC = ∠BCD = ∠CDE = ∠DEA; [III. 27.

i.e. fig. ABCDE is also equiangular.

∴ a regular pentagon ABCDE has been inscribed in the given ⊙.

Cor.—**The circumference of a circle can be divided into 5, 10, 20, 40, . . . equal arcs.**

Ex. 618.—An equiangular pentagon inscribed in a circle is equilateral.

Let ABCDE (fig. of Prop. 11) be an equiangr. pentagon, then it is equilateral.

∵ ∠A = ∠B;

∴ arc BCDE = arc AEDC,

Take away common arc CDE,

then arc BC = arc AE;

∴ chd. BC = chd. AE.

Similarly chd. AE = chd. CD

= chd. AB

= chd. ED

(each side = the next but one to it);

∴ the fig. is equilateral.

The method of proof is quite general.

In any equiangular figure inscribed in a circle each side is equal to the next but one to it.

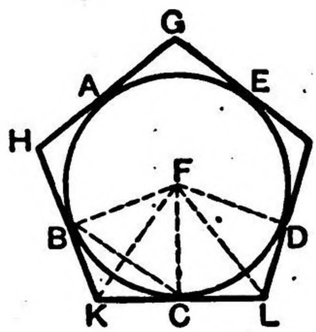
Hence :—**If an equiangular figure inscribed in a circle have an odd number of sides it must be equilateral.**

PROPOSITION 12. PROBLEM.

About a given circle to describe a regular pentagon.

Let ABCDE be the given ⊙; it is reqd. to describe a regr. pentagon about it.

Let A, B, C, D, E be the angular pts. of a regr. pentagon inscribed in the given ⊙. [IV. 11.



At A, B, C, D, E draw tangents forming the closed fig. GHKLM: this shall be the pentagon reqd.

Find F the centre of the ⊙. Join FB, FK, FC, FL, FD, BC.

Then ∠ KBC = ∠ in alt. segt. cut off by BC
= ∠ KCB; } III. 32.
∴ KB = KC.

In △s FKB, FKC,
FK, KB, BF = FK, KC, CF respectively;
∴ ∠ BFK = ∠ CFK,
and ∠ BKF = ∠ CKF,

i.e. ∠ CFK is half of ∠ BFC,
and ∠ CKF is half of ∠ BKC.

Similarly ∠ CFL is half of ∠ CFD,
and ∠ CLF is half of ∠ CLD.

But ∠ BFC = ∠ CFD (∵ arc BC = arc CD);
∴ ∠ CFK = ∠ CFL.

Also in △s CFK, CFL,
rt. ∠ FCK = rt. ∠ FCL,
and CF, opp. equal ∠ s in each, is common ;
∴ ∠ CKF = ∠ CLF,
and KC = CL,
i.e. KL is double of KC.

Similarly HK is double of BK;
∴ HK = KL (∵ BK = KC).

Similarly KL = LM = MG = GH,
i.e. GHKLM is equilateral.

Again,

∵ ∠ HKL is double of CKF,
and ∠ KLM is double of ∠ CLF,
and ∠ CKF = ∠ CLF;
∴ ∠ HKL = ∠ KLM.

Similarly ∠ KLM = ∠ M = ∠ G = ∠ H,
i.e. GHKLM is also equiangular,
∵ it is the reqd. regr. pentagon about ⊙ ABCDE.

Alternative Proof.—(With the same construction.)

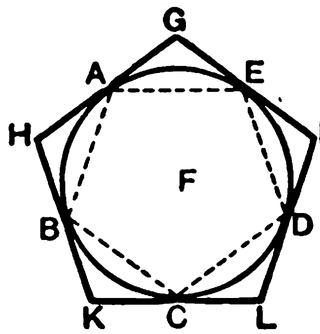
Join AB, BC, CD, DE, EA.

∵ arc AB = arc BC,

∴ ∠ in segt. AEDCB = ∠ in segt. BAEDC. [III. 27.

But ∠ HAB = ∠ in segt. AEDCB (∵ HA touches the ⊙) } III. 32.
and ∠ KBC = ∠ in segt. BAEDC (∵ KB touches the ⊙) }

∴ ∠ HAB = ∠ KBC.



Similarly ∠ HBA = ∠ KCB.
Also AB = BC,
∴ HB = KC,
and ∠ H = ∠ K.
Similarly BK = CL;
∴ whole HK = whole KL.
Similarly KL = LM,
= MG,
= GH;
∴ the fig. is equilateral.

Also, as we have proved, ∠ H = ∠ K, so we can prove that
∠ K = ∠ L,
= ∠ M
= ∠ G,
and ∴ that the fig. is equiangular;
∴ fig. GHKLM is the reg. pentagon reqd.

NOTE.

The word 'pentagon' does not occur in our demonstration of the regularity of the figure. The method is therefore general, and if we can inscribe in a circle a regular polygon of some given number of sides, we can therefore also circumscribe about a circle a regular polygon of the same number of sides.

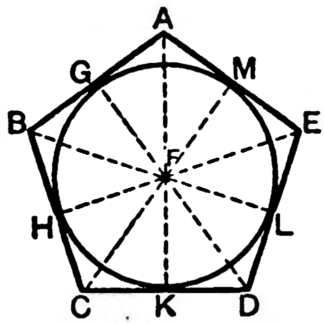
We prove any side equal to the next and any angle equal to the next; hence the figure is equilateral and equiangular.

PROPOSITION 13. PROBLEM.

To inscribe a circle in a given regular pentagon.

Let ABCDE be the given regular pentagon; it is reqd. to describe a ⊙ in it.

Bisect ∠s BCD, CDE by CF, DF. Join FB, FA, FE.



In △s BCF, DCF, BC, CF = DC, CF,
and ∠BCF = ∠DCF;
∴ ∠CBF = ∠CDF.
But ∠CDF is half of ∠CDE, which = ∠CBA;
∴ ∠CBF is half of ∠CBA.
Similarly, FA, FE bisect ∠s BAE, AED.
Now drop ⊥rs FG, FH, FK, FL, FM on the sides AB, BC, CD, DE, EA.
In △s FHC, FKC, ∠FCH = ∠FCK,
the rt. ∠FHC = rt. ∠FKC, and CF is common;
∴ FH = FK.

Similarly FK = FL = FM = FG.

∴ the ⊙ with centre F and rad. FH will pass through G, H, K, L, M, and will touch the sides of the pentagon (∵ the ∠s at G, H, K, L, M are rt. ∠s);

∴ it is the ⊙ reqd.

Note that for 'pentagon' the word 'polygon' might be substituted without invalidating the proof.

We have therefore a general method of inscribing a circle in a given regular polygon of any number of sides. Compare IV. 8.

Ex. 619.—**An equilateral pentagon described about a circle is equiangular.**

Let ABCDE be an equilateral pentagon described about the ⊙ GHM (fig. of IV. 13), whose centre is F.

Join FH, FB, FG, FA, FM, FE.
Then AG = AM;
but AB = AE;
∴ GB = ME.

In △s FGB, FEM $\begin{cases} \text{FG, GB} = \text{FM, ME,} \\ \text{and rt. } \angle \text{FGB} = \text{rt. } \angle \text{FME;} \end{cases}$
∴ ∠FBG = ∠FEM.

But ∠s ABC, AED are easily shown to be double of ∠s FBG, FEM. (See Note 3, p. 193.)
∴ ∠ABC = ∠AED.
Similarly ∠AED = ∠DCB,
= ∠BAE,
= ∠EDC,
(each ∠ = the next but one to it).
∴ the fig. is equiangular.

The method of proof is quite general.

In any equilateral figure circumscribed about a circle, each angle is equal to the next but one to it.

Hence :—

If an equilateral figure circumscribed about a circle have an odd number of sides, it must be equiangular. Compare Ex. 618.

Ex. 620.—**If an equiangular pentagon be described about a circle, it must be equilateral.**

Use the fig. of IV. 13. The quadrilaterals AGFM, BHFG, etc., are cyclic. Hence ∠s MFG, GFH, etc., each = ext. angle of given pentagon. Hence theorem follows by IV. 12.

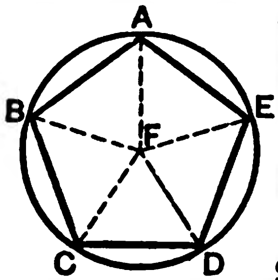
Note that the theorem is general.

PROPOSITION 14. PROBLEM.

To describe a circle about a given regular pentagon.

Let ABCDE be the given pentagon ; it is reqd. to describe a ⊙ about it.

Bisect ∠s BCD, CDE by CF, DF.



Join FB, FA, FE.

In △s BCF, DCF $\begin{cases} \text{BC, CF} = \text{DC, CF,} \\ \text{and } \angle\text{BCF} = \angle\text{DCF,} \end{cases}$

∴ ∠CBF = ∠CDF.

But ∠CDF is half of CDE, which = ∠ABC,

∴ ∠CBF is half of ∠ABC.

Similarly FA, FE bisect ∠s EAB, DEA.

Again ∵ ∠BCD = ∠CDE,

∴ their halves are equal ;

i.e. ∠FCD = ∠FDC ;

∴ FC = FD.

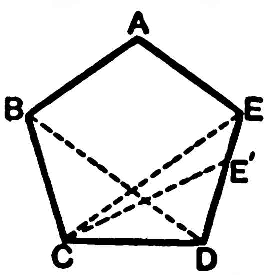
Similarly it may be shown that FD = FE = FA = FB ;

∴ the ⊙ described with centre F and radius FA will pass through A, B, C, D, E ;

∴ it is the ⊙ reqd.

Alternative Method.

Join BD, and about △BCD describe a ⊙ : this shall be the ⊙ reqd.



For if it does not pass through E, let it cut DE or DE produced in E′. Join CE, CE′.

In △s CDE, BCD $\begin{cases} \text{CD, DE} = \text{BC, CD,} \\ \text{and } \angle\text{CDE} = \angle\text{BCD,} \end{cases}$

∴ ∠CED = ∠CBD.

= ∠CE′D in the same segt.,

which is impossible. [I. 16.

Similarly it can be shown that the ⊙ will pass through A.

Note that each of the above methods is general.

We have therefore two general methods of describing a circle about a given regular polygon of any number of sides.

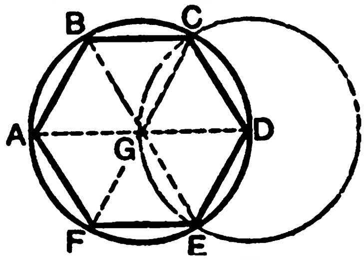
Ex. 621.—The circle through any three successive vertices of a regular polygon passes through all the remaining vertices.

Ex. 622.—The circle through any three vertices of a regular polygon passes through the remaining vertices.

PROPOSITION 15. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given ⊙.
Find the centre G, and draw the diamr. AGD.
With centre D and radius DG describe a ⊙ EGC.
Draw the diamrs. CGF, EGB.
Join AB, BC, CD, DE, EF, FA.
ABCDEF shall be the reqd. hexagon.



∵ △CGD is equilateral, [I. 1.
∴ all its angles are equal.
∴ ∠CGD = one-third of two rt. ∠s.
Similarly ∠EGD = one-third of two rt. ∠s.
∴ ∠CGB = one-third of two rt. ∠s, [I. 13.
[∵ BGE is a st. line];
∴ ∠CGD = ∠EGD = ∠CGB.
But ∠CGD = ∠AGF, ∠EGD = ∠BGA,
and ∠CGB = ∠EGF;
∴ these six ∠s are equal;
∴ the arcs AB, BC, CD, DE, EF, FA are equal;
∴ the chds. AB, BC, CD, DE, EF, FA are equal;
i.e. the hexagon is equilateral.
Also each of its angles stands on an arc made up of four of the six equal arcs AB, BC, CD, DE, EF, FA.
∴ it is also equiangular;
∴ it is the hexagon reqd.

Cor. (i.).—From this it is clear that the side of the hexagon is equal to the radius of the circle.

Also that the methods of IV. 12, IV. 13, IV. 14 are as applicable to the regular hexagon as to the regular pentagon.

Cor. (ii.)—The circumference of a circle can be divided into 3, 6, 12, 24 . . . equal arcs.

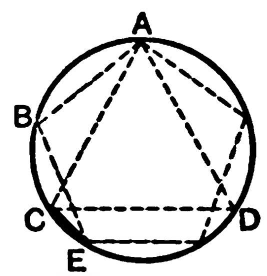
Ex. 623.—To describe a regular hexagon on a given finite straight line.

PROPOSITION 16. Problem.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABC be the given ⊙, AC the side of an equilateral △, and AB, BE sides of a regular pentagon inscribed in it.

Then of such equal arcs as the whole ○ce contains (fifteen), the arc ABC (which is one-third of it) contains five, and the arcs AB, BE (which are each one-fifth of it) contain six;



∴ the arc CE is one-fifteenth part of the ○ce;

∴ if chds. equal to CE be placed round in the whole ⊙, a regular quindecagon will be described.

The methods of IV. 12, IV. 13, and IV. 14 apply to the regular quindecagon as well as to the regular pentagon;

NOTE.

Euclid does not prove that the quindecagon is equiangular. This is easily deduced from the fact that each angle stands on an arc which is made up of thirteen of the fifteen equal arcs subtended by the sides.

Compare the demonstrations of IV. 11 and IV. 15, and note that if a figure inscribed in a circle be equilateral, it must be equiangular, since each of its angles stands on an arc which is equal to the whole circumference diminished by two of the whole set of equal arcs subtended by the sides of the figure.

ON THE CIRCUMSCRIPTION AND INSCRIPTION OF CIRCLES.

'The locus of the centre of a circle **which passes through two given points** is **the perpendicular bisector of their join**' (see p. 165).

Hence, *when a polygon is such that a circle can be described about it* the centre of that circum-circle **is the point of intersection of any two perpendicular bisectors of its sides which are not coincident.**

We can therefore determine the centre by drawing the perpendicular bisectors of any two consecutive sides.

'The locus of the centre of a circle **which touches two straight lines drawn from a point** is the internal bisector of the angle between them' (see p. 193).

Hence, *when a polygon is such that a circle can be inscribed in it* the centre of that in-circle is the point of intersection of **any two internal bisectors of its angles which are not coincident.**

We can therefore determine its centre by drawing the internal bisectors of any two consecutive angles.

The student is requested to notice that, speaking generally,

(1) to get the centres of **circum-circles** we bisect **sides.**

(2) to get the centres of **in-circles** we bisect **angles.**

Regular figures have the exceptional property that the point of intersection of the perpendicular bisectors of any two consecutive sides coincides with that of the internal bisectors of any two consecutive angles, and we have, for the sake of uniformity, bisected angles to get the centres of the circum-circles of the square (IV. 9), and regular pentagon (IV. 14), as well as those of the in-circles (IV. 8, 13). The same method would apply to the circum-circle of the regular (*i.e.* the equilateral) triangle, *but not to that of any figure* (*triangle or other*) *which is irregular.*

It will be found on examination that a regular polygon of n sides (sometimes called a regular **n-gon**) has n axes of symmetry which all pass through a common point equidistant from the vertices and also from the sides, and which is therefore both the circum-centre and the in-centre of the polygon.

If n is odd each axis passes through one vertex and through the mid-point of one side.

If n is even the axes are of two kinds; one half of them pass each through two vertices, the other half each through the mid-points of two sides. (Henrici, *Congruent Figures.*)

We can determine the common centre of the in-circle and circum-circle by drawing *any two* of these axes of symmetry.

In the case of the square the simplest *practical* way is to draw the two diagonals (which are axes of symmetry). A similar method applies to all regular polygons of an even number of sides.

This common centre is sometimes called the centre of the polygon.

Ex. 624.—ABCDEF is a regular hexagon; show that AD is an axis of symmetry (see p. 23), and hence determine its in-centre.

Ex. 625.—Apply a similar treatment to a regular octagon ABCDEFGH.

In the case of a regular pentagon there is no simpler way than that given in the text for determining two axes of symmetry, viz. that of drawing the internal bisectors of two of its angles, and the same remark applies to any regular figure with an odd number of sides.

It is worth noticing that

(1) Any *equilateral* figure *inscribed in* a circle is *equiangular.*

(2) Any *equiangular* figure *described about* a circle is *equilateral.*

But that we can only assert that

(3) Any *equiangular* figure *inscribed in* a circle is *equilateral if the number of sides be odd.*

(4) Any *equilateral* figure *described about* a circle is *equiangular if the number of sides be odd.*

To fix these four propositions in the mind it is best to take *the simplest illustrative cases* of the necessity in (3) and (4) of the restriction that the number of sides must be odd.

If we join the ends of *any two diameters* of a circle we obtain a *rectangle* which is *equiangular* although *not necessarily equilateral.*

If we draw tangents at the ends of *any two diameters* of a circle we obtain a *rhombus* which is *equilateral* although *not necessarily equiangular.*

See Exs. 172, 173.

DEFINITIONS.

Book IV.

I. A rectilineal figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

II. In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

III. A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

IV. A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

V. In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI. A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

VII. A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

MISCELLANEOUS EXERCISES.

(Book IV.)

Ex. 626.—Any odd number of equal arcs AB, BC, . . . GH being taken consecutively on a circle, show that AH is parallel to one of the equal chords subtending them.

Ex. 627.—Any even number of equal arcs AB, BC, . . . GH being taken on a circle, show that AH is parallel to the tangent at one of the points of section.

Ex. 628.—Each diagonal of a regular polygon of an odd number of sides is parallel to one of the sides, and to the tangent at one of the vertices.

Ex. 629.—Each diagonal of a regular polygon of an even number of sides is parallel either to two of the sides or to the tangents at two of the vertices.

Ex. 630.—To describe a regular hexagon on a given straight line.

Ex. 631.—The regular hexagon inscribed in a circle is three-fourths of that circumscribed about the same circle.

Ex. 632.—The circum-radius of one regular hexagon is twice the in-radius of another; show that the first is three times the second in area.

Prove, by dissection and superposition, that the first contains nine of such equal parts as the second is divided into by perpendiculars from its centre on three alternate sides.

Ex. 633.—AB is the diameter of a circle; AD, DE sides of a regular inscribed pentagon; C any point on the circumference. Show that angle BCE is one-fifth of a right angle.

Ex. 634.—AB, BC, CD are three equal arcs of a $\odot$ABCD; AE, EBF, FCG, GD are tangents at A, B, C, D. Show that AC, BD, EG are concurrent.

Ex. 635.—DA is one side of a regular hexagon inscribed in a circle; AB is a tangent equal to AD and making an obtuse angle with it. If BD cut the circle in E, show that AE is the side of a regular duodecagon in the same circle.

Also if the line joining B to the centre cut the circle in F, EF is the side of a regular 24-gon in the same circle.

Ex. 636.—Regular pentagons ABC′D′E′, aBCde are constructed on the sides AB, BC of a given regular pentagon. Show that BC, Bd, Be are in a straight line with BD′, BE′, BA respectively.

Ex. 637.—Show how to cut out six equal regular pentagons from a given regular pentagon.

Ex. 638.—A, B, C, D, E, F, G, H, K, L are points which divide a circle whose centre is O into ten equal arcs.

If BG, AD intersect in M show (from III. 27, Cor.) that

$$\angle AMB = \angle GBA = \angle FAB$$

and that $\angle AOB = \angle OAM = \angle BAM$.

Show also that rect. OB . BM = sq. on OM, and deduce the construction of IV. 10.

Ex. 639.—H is the mid-point of the side AB of a regular n-gon whose centre is C. If HC is produced to D so that CD = CA or CB, show that D is the centre of a regular 2n-gon whose side is AB.

Hence if R, r be the circum- and in-radius of a regular polygon of any number of sides, R′, r′ those of a regular polygon of the same perimeter but double the number of sides,

$$R'^2 = \tfrac{1}{2} R(R+r)$$
$$r' = \tfrac{1}{2}(R+r).$$

(Legendre, *Éléments de Géométrie.*)

THE PRINCIPAL CIRCLES OF A TRIANGLE.

1. The **Circum-circle.**

2. The **In-circle.**

3. The **Ex-circles.**

4. The **Nine-point-circle.**

5. The **Tucker-circles** (including as special cases the **Cosine, Triplicate Ratio,** and **Taylor** circles).

6. The **Brocard circle.**

The first three of these have been discussed at length in IV. 4, 5, and the notes appended thereto.

The theorem involved in the definition of the fourth—the nine-point-circle—has been given as Ex. 434; the demonstration of the theorem being suggested to the student by the previous chain of exercises, 427-433.

On account of its remarkable properties, and the attention paid to it by geometers, we add some demonstrations of the property from which it derives its name.

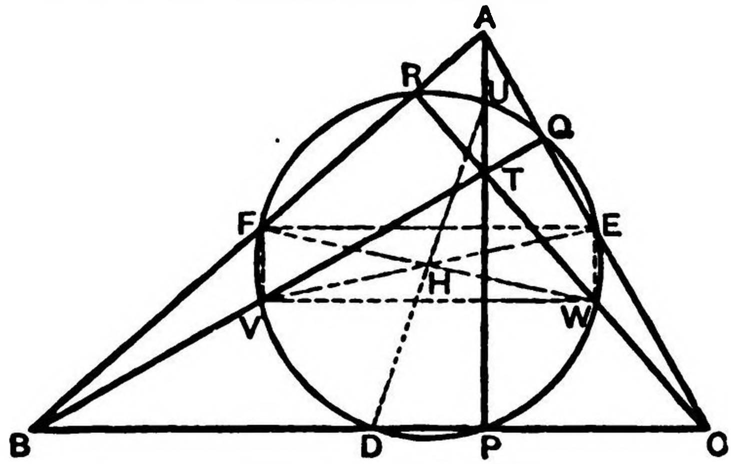
PROPOSITION.

The projections P, Q, R of the vertices A, B, C of a triangle ABC on the opposite sides BC, CA, AB; the mid-point D, E, F of those sides; and the mid-points U, V, W of the lines TA, TB, TC joining the orthocentre T to the vertices, are concyclic.

First Demonstration.

Join EF, FV, VW, WE, EV, FW.
∵ F, V are mid-pts. of AB, TB;
∴ FV is ‖ to AT.
Similarly EW is ‖ to AT, and EF, VW are ‖ to BC.
But AT is ⊥r to BC;
∴ EFVW is a rectangle;
∴ its diagls. EV, FW are equal and bisect each other.

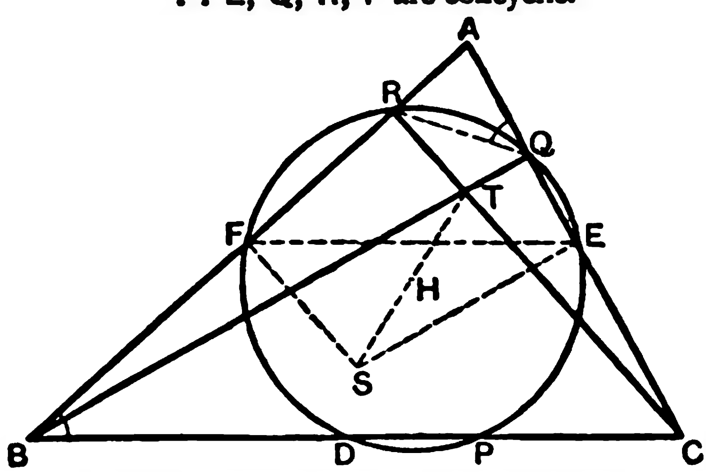
Similarly FW, DU are equal and bisect each other ;
∴ DU, EV, FW are all equal and bisect each other ;
∴ the ⊙ on DU as diamr. will pass through E, F, V, W.



Also ∵ DPU is a rt. ∠
it will pass through P.
Similarly it will pass through Q and R.

SECOND DEMONSTRATION.

Let S be the circum-centre of △ABC,
join QR, EF, SE, SF, ST and bisect ST in H.
∵ BRC, BQC are rt. ∠s,
∴ B, R, Q, C are concyclic ;
∴ ∠AQR = ∠ABC
= ∠AFE (∵ EF is ‖ to BC);
∴ E, Q, R, F are concyclic.



Now the perpr. bisectors of EQ, FR must each pass through the mid-pt. H of ST (∵ SE, TQ are ⊥r to EQ and SF, TR to FR) ;
∴ H is the centre of a ⊙ passing through E, Q, R, F.

Similarly H is the centre of a ⊙ passing through F, R, P, D;
∴ these ⊙s must coincide, [III. 5.
i.e. the ⊙ through P, Q, R passes through the mid-pts. D, E, F of BC, CA, AB.

But P, Q, R are also projns. of the vertices of △ ATB on the opposite sides;
∴ the ⊙ through P, Q, R passes through the mid-pts. U, V of TA, TB.
Similarly it can be shown to pass through W.

Third Demonstration.

The construction of the figure is left to the student.

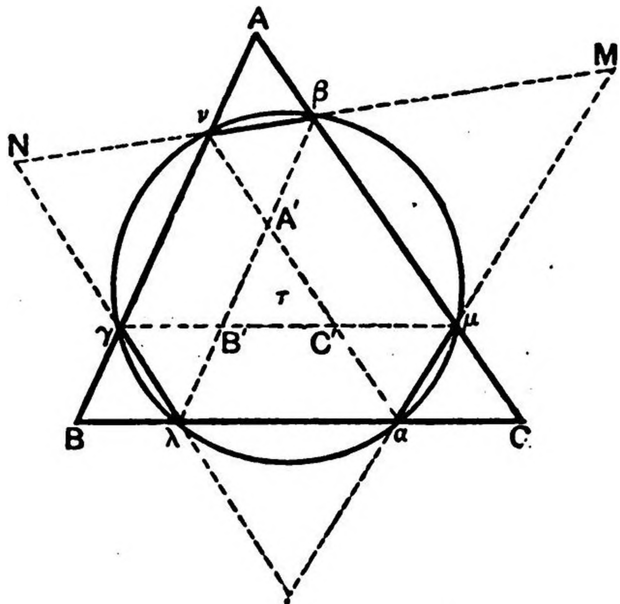
Join QU, QR, QD, RU, RD,
∵ AQT, ART are rt. ∠s;
∴ A, Q, T, R lie on a ⊙ whose centre is U.
Hence ∠ URT = ∠ UTR (∵ rad. UR = rad. UT)
= ∠ CTP.
Similarly C, Q, R, B lie on a ⊙ whose centre is D,
and ∠ DRC = ∠ DCR (∵ rad. DC = rad. DR);
∴ whole ∠ DRU = ∠s CTP, DCR,
= ext. ∠ DPT, which is a rt. ∠.
Similarly ∠ DQU is a rt. ∠;
∴ the ⊙ on DU as diamr. passes through P, Q, R,
i.e. the ⊙ through P, Q, R has DU for a diamr.
Similarly it has EV and FW for diamrs.

The previous circles and their most important properties have long been known to geometers. We have now to draw the attention of the student to a group of circles whose existence seems to have passed unnoticed until quite recently. In the demonstrations given of their properties, we shall suppose the triangle ABC neither *right-angled* nor *isosceles*.

ANTI-PARALLELS—TUCKER'S CIRCLES.

Def.—**If a straight line, $\beta\nu$, makes with the side AC of a triangle ABC the angle $A\beta\nu$, equal to the angle B (and therefore also the angle $A\nu\beta$ with AB equal to the angle C) it is called 'an anti-parallel to BC with respect to A.'**

When there is no doubt with respect to what vertex the anti-parallelism exists, the words 'with respect to A' are frequently omitted.



When a 'parallel' or an 'anti-parallel' to the base of a triangle is mentioned, the student must gather from the context whether an indefinite straight line is intended or the segment of that straight line intercepted between the other two sides of the triangle. If we speak of its 'ends,' or its 'mid-points,' we are of course using the word in its restricted sense.

Prop. I.—**All anti-parallels to a given side of a triangle with respect to the opposite vertex are parallel to one another and to the tangent at that vertex to the circum-circle of the triangle.** See Ex. 362.

Conversely:—**All parallels to the tangent at a vertex to the circum-circle of the triangle are anti-parallel to the opposite side with respect to that vertex.**

PROP. 2.—**The ends of any side of a triangle and an anti-parallel to it with respect to the opposite vertex are concyclic.**

PROP. 3.—**The ends of a parallel and an anti-parallel to the same side of a triangle are concyclic.**

PROP. 4.—**If the anti-parallels $\beta\nu$, $\lambda\gamma$ to BC, CA are equal, then $\beta\lambda$ is parallel to AB.**

Conversely :—**If $\beta\lambda$ is parallel to AB, the anti-parallels $\beta\nu$, $\lambda\gamma$ to BC, CA are equal.**

(*Since* $\triangle$ ABC *is not right angled, the anti-parallels will meet: let them meet in* N, *and show that* $\triangle$*s* $N\nu\gamma$, $N\beta\lambda$ *are isosceles.*)

COR.—**Given any anti-parallel $\beta\nu$ to BC within a triangle ABC, two anti-parallels $\gamma\lambda$, $\alpha\mu$ to CA, AB equal to $\beta\nu$, can always be found within the triangle.**

(*By drawing parallels* $\beta\lambda$, $\nu\alpha$ *to* AB, AC.)

PROP. 5.—**If $\beta\nu$, $\gamma\lambda$, $\alpha\mu$ be three equal anti-parallels to BC, AC, AB, their six ends β, ν, γ, λ, α, μ are concyclic.**

Since $\triangle$ ABC is not right angled, no two of these anti-parallels can be parallel to each other. They will therefore in general form a triangle LMN.

Let τ be its in-centre.

$\because$ $\beta\nu$, $\gamma\lambda$ are equally inclined to AB ;

$\therefore$ $N\nu\gamma$ is isosceles ;

$\therefore$ the internal bisector of $\angle$ N bisects $\gamma\nu$ at rt. $\angle$s ;

$\therefore$ the $\perp$r bisector of $\nu\gamma$ passes through τ.

Similarly the $\perp$r bisector of $\lambda\alpha$ passes through τ

Again $\because$ $\beta\nu = \alpha\mu$,

$\therefore$ $\nu\alpha$ is $\parallel$ to AC ;

$\therefore$ ν, α, λ, γ are concyclic ;

$\therefore$ τ is the centre of a $\odot$ through ν, α, λ, γ.

Similarly τ is the centre of a $\odot$ through β, λ, α, μ ;

$\therefore$ the six points β, ν, γ, λ, α, μ all lie on one and the same $\odot$.

Such a circle is called a 'Tucker circle' of the triangle ABO.

COR.—If any two of the three points L, M, N coincide, the three coincide, and the point of coincidence K is the centre of the circle through β, ν, γ, λ, α, μ, which is then called the 'cosine circle' of the triangle ABC.

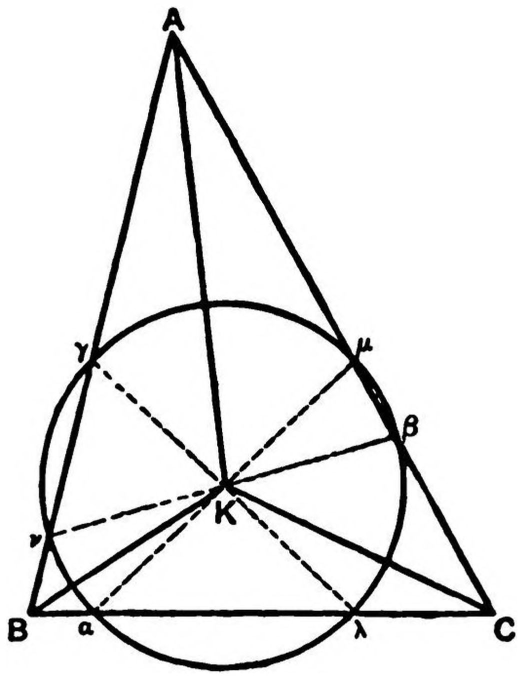
SYMMEDIANS — COSINE, TRIPLICATE RATIO, AND TAYLOR'S CIRCLES.

Prop. 6.—**The locus of the mid-points of anti-parallels to a side of a triangle with respect to the opposite vertex is a straight line.**

The proof is left as an exercise to the student. (Compare Ex. 109.)

Such a straight line is called a 'Symmedian line,' or a 'Symmedian' of the triangle.

Prop. 7.—**The three symmedian lines of a triangle co-intersect at the centre of a Tucker's Circle.**



Through the intersection K of the symmedians through B and C, draw anti-parallels $\beta\nu$, $\gamma\lambda$, $\alpha\mu$ to BC, CA, AB.

Then $K\beta = K\mu$ ($\because \angle K\beta\mu = \angle B = \angle K\mu\beta$)

$= K\alpha$ ($\because$ K is on the symn. through C)

$= K\lambda$ ($\because \angle K\alpha\lambda = \angle A = \angle K\lambda\alpha$)

$= K\gamma$ ($\because$ K is on the symn. through B)

$= K\nu$ ($\because \angle K\gamma\nu = \angle C = \angle K\nu\gamma$);

$\therefore$ (1) K is on the symmedian through A,

(2) a circle can be described with centre K to pass through the ends of the equal anti-parallels $\beta\nu$, $\gamma\lambda$, $\alpha\mu$.

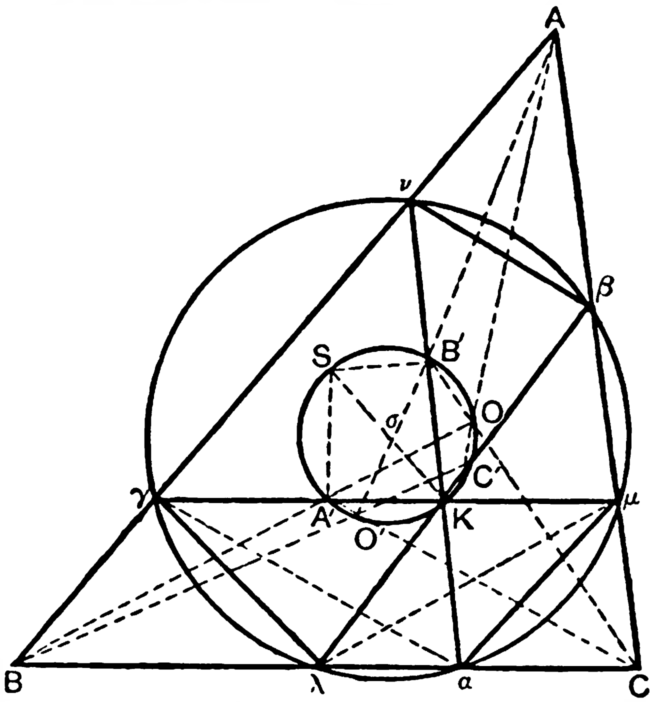
(i) This point K is called **the Symmedian point** of the triangle ABC.

It is also sometimes spoken of as '**Lemoine's point**,' or the '**point de Grebe.**'

(ii) This circle is called the **Cosine circle** of the triangle ABC.

It is also sometimes spoken of as the **second Lemoine circle.**

PROP. 8.—**The ends of the parallels $\mu\gamma$, $\nu\alpha$, $\beta\lambda$ to the sides BC, CA, AB of a triangle ABC through the symmedian point K lie on a Tucker's circle whose centre is the mid-point of the line joining K to the circum-centre S of triangle ABC.**



$\because$ $A\nu K\beta$ is a $\|$gm,

$\therefore$ KA bisects $\beta\nu$;

$\therefore$ $\beta\nu$ is anti-parallel to BC.

Similarly $\gamma\lambda$, $\alpha\mu$ are anti-parallel to CA, AB. Also $\beta\nu=\gamma\lambda=\alpha\mu$.

Take A′ on $\mu\gamma$ such that

$$\gamma A' = K\mu = \alpha C,$$

and $\therefore$ $\mu A' = K\gamma = B\lambda$;

$\therefore$ $C\alpha\gamma A'$, $B\lambda\mu A'$ are $\|$gms,

and KA′, $\mu\gamma$ have a common perpr. bisector.

$$\text{Then } BA' = \lambda\mu$$
$$= \alpha\gamma$$
$$= A'C.$$

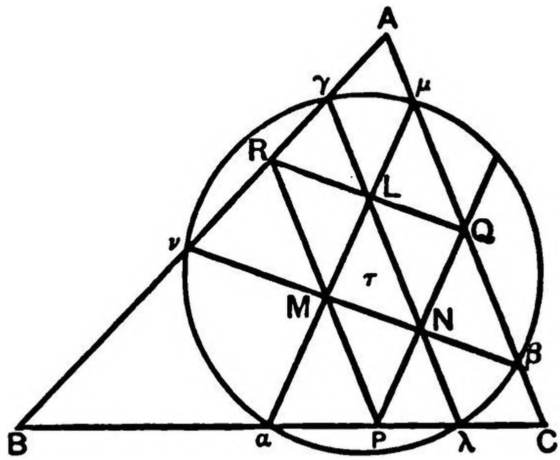
$\therefore$ the circum-centre S lies on the perpr. to BC through A′.

Again, the perpr. bisector of $\alpha\lambda$ is also that of $\mu\gamma$, and $\therefore$ of A'K;
$\therefore$ it passes through σ, the mid-pt. of KS.

Similarly the perpr. bisector of $\beta\mu$ also passes through σ,
$\therefore$ σ is the centre of the ⊙ through α, μ, β, λ.

Similarly it is the centre of the ⊙ through β, ν, γ, μ,
$\therefore$ α, μ, β, ν, γ, λ lie on a ⊙ whose centre is σ.

This circle is called the **Triplicate Ratio Circle** of the triangle ABC. It is also sometimes spoken of as the **First Lemoine Circle.**



PROP. 9.—**If $\beta\nu$, $\gamma\lambda$, $\alpha\mu$ be the anti-parallels obtained by joining the mid-points L, M, N of the sides of the pedal triangle PQR of the triangle ABC, their ends lie on a Tucker's circle whose centre is the in-centre, τ, of triangle LMN.**

$\because$ LM, LN are $\parallel$ to PQ, PR they are equally inclined to BC,

$\therefore$ $L\alpha = L\lambda$. (1)

Again, $\because$ $L\mu$ is $\parallel$ to PQ,

$L\mu$, LQ are equally inclined to AC,

$\therefore$ $L\mu = LQ$.

Similarly $L\gamma = LR$,

$\therefore$ $L\mu = L\gamma$. (2)

From (1) and (2) it follows that the internal bisector of $\angle$ NLM is the common perpr. bisector of $\alpha\lambda$, $\mu\gamma$ and that the anti-parallels $\gamma\lambda$, $\alpha\mu$ are equal.

Similarly for the perpr. bisector of $\beta\mu$, $\nu\alpha$; and for that $\gamma\nu$, $\lambda\beta$,
$\therefore$ these bisectors cointersect at the in-centre τ of $\triangle$ LMN

And $\beta\nu = \gamma\lambda = \alpha\mu$.

Hence the theorem.

Cor.—The intersections of the circle with the sides of the triangle are the projections of P, Q, R on the sides.

$$\because L\mu = LQ = L\gamma = LR,$$

$\therefore$ the $\odot$ on RQ as diamr. passes through μ, γ,

$\therefore$ $R\mu Q$, $Q\gamma R$ are rt. $\angle$s.

This circle is sometimes called **Taylor's Circle.**

THE BROCARD POINTS AND THE BROCARD CIRCLE.

PROP. 10.—**One point O and only one can be found within a triangle such that** $\angle OAB = \angle OBC = \angle OCA$.

It has been shown on p. 221 that such a point exists.

To show that there is only one, let O be any such pt.; describe a ⊙ about △ AOC; produce BO to cut in P, and join AP.

Then ∵ $\angle OAB = \angle OCA$,
the ⊙ touches AB at A,
∴ it is a fixed ⊙.
Also $\angle OPA = \angle OCA$,
$= \angle OBC$,
∴ AP is ∥ to BC;
∴ P is a fixed pt.;
∴ BP is a fixed st. line;
and ∴ O a fixed pt.

PROP. 11.—**One point O′, and only one, can be found within a triangle ABC such that** $\angle O'BA = \angle O'CB = \angle O'AC$.

PROP. 12.—**The circum-centre S, the symmedian point K, and the Brocard points O, O′ of a triangle ABC are concyclic.**

On the parallels $\mu\gamma$, $\nu\alpha$, $\lambda\beta$ through K to BC, CA, AB take A′, B′, C such that

$\gamma A' = K\mu$; $\alpha B' = K\nu$; $\beta C' = K\lambda$. [Fig. of Prop. 8.

Then it follows, as in Prop. 8, that SA′ is ⊥r to $\mu\gamma$, and $\mu\lambda$ is ∥ to BA′;
∴ the ⊙ on KS as diamr. passes through A′.
Similarly it passes through B′ and C′.

From this circle cut off segts. KSO, KSO′, each containing an angle equal to the angle ω, subtended at the circumference of the ⊙ through α, μ, β, ν, γ, λ by one of the equal anti-parallels $\alpha\mu$, $\beta\nu$, $\gamma\lambda$.

Then $\angle KA'O = \angle \mu\lambda\alpha$,
$= \angle CBA'$,
∴ B, A′, O are in a st. line;
∴ $\angle OBC = \omega$.
Similarly $\angle OCA = \omega = \angle OAB$,
∴ O is a Brocard point of △ ABC.
Similarly O′ is a Brocard point of △ ABC.

COR.—A′, B′, C′ are the intersections of OB, OC, OA with O′C, O′A, O′B respectively.

The circle on KS as diameter is called **the Brocard circle** of the triangle ABC.

The angle ω is called **the Brocard angle** of the triangle ABC.

For further information on this branch of Geometry the student is referred to Casey's *Sequel to Euclid*; to the article by the Rev. T. C. Simmons on *The Recent Geometry of the Triangle* in Milne's *Companion to Problem Papers*, and to a paper by R. F. Davis, M.A., in the Fourteenth Report of the Association for the Improvement of Geometrical Teaching.

Ex. 640.—A circle is drawn to pass through the mid-point D of the base BC of a triangle ABC and touch AB at A. Show that it cuts BC again at the same point as the circle drawn through C to touch the symmedian through A at A.

Show also that the above is a general property for pairs of *isogonal lines* (see p. 236).

Ex. 641.—The circle through the mid-point of BC touching AB at A cuts the join of the mid-points of AB, BC at the same point as the symmedian through A.

Ex. 642.—The circles touching the symmedian through A and passing respectively through B and C are equal.

Ex. 643.—In the figure of IV. 10 CD is an anti-parallel to AD with respect to B.

Ex. 644.—In any triangle ABC, CD is drawn anti-parallel to AC with respect to B; show that BC touches the circum-circle of triangle CAD.

Ex. 645.—ABCD is a cyclic quadrilateral. If AC is a symmedian of triangle ABD, show that it is also a symmedian of triangle BCD, and that BD is a symmedian of each of the triangles ABC, ACD.

Ex. 646.—In an isosceles triangle each Brocard point lies on a median.

Ex. 647.—If O, O′ are the Brocard points of triangle ABC, as defined on p. 221, show that the circles AOC, AO′B intersect on the symmedian through A.

Show also that they intersect on the Brocard circle of triangle ABC.

MISCELLANEOUS EXERCISES.

(Books I.-IV.)

Ex. 648.—AOB, COD are two intersecting straight lines, and each of the figures AOCE, BODF is a rhombus. Show that EF passes through O, and that AC, BD are parallel.

Ex. 649.—Straight lines drawn through A, C, the extremities of one diagonal of a parallelogram ABCD, respectively perpendicular and parallel to the other diagonal, intersect in E. Prove that BE = CD.

Ex. 650.—ABC is a triangle having an acute angle B which is greater than the angle at A: the side AB is produced to D, and BE is drawn to meet AC produced in E in such a way that the angle DBE is equal to the angle ABC. Show that BE is longer than BC.

Ex. 651.—If one of the parallelograms about the diagonal of any parallelogram is equal to half one of the complements, show that the complement is equal to half the other parallelogram.

Ex. 652.—Divide a given straight line into two parts so that the rectangle contained by the whole and one part may be equal to the rectangle contained by the other part and a given finite straight line.

Ex. 653.—The diagonal AC of a square ABCD is produced to E, so that CE = BC. Prove that square on BE = rectangle AC, AE.

Ex. 654.—ABC is a triangle having a right angle at C: from any point D in AC is drawn DE perpendicular to AB. *Without using any property of the circle*, show that rectangle CA.AD = rectangle BA.AE.

Ex. 655.—A line of given length moves with its ends on two straight lines at right angles to each other. Find when its mid-point is at (1) its minimum, (2) its maximum distance from a given straight line, or a given circle.

Ex. 656.—The circle described with centre A and radius AB cuts the circum-circle of the rectangle ABCD in E. Show that CE = AD, and that DE is parallel to AC.

Ex. 657.—If one of two equal chords of a circle bisects the other, then each bisects the other.

Ex. 658.—With the notation of Ex. 426-439, show that the circum-circles of the cyclic quadrilaterals AQTR, BRQC cut each other orthogonally. Similarly for the circum-circles of CPTQ, AQPB, and of BRTP, CPRA.

Ex. 659.—ABCD is a quadrilateral having CD parallel to AB. If AD is a tangent to the circum-circle of triangle ABC, show that BC is a tangent to the circum-circle of triangle ACD.

Ex. 660.—Show that all circles whose centres lie on a given straight line, and whose circumferences pass through a given point, have a common chord of intersection.

Ex. 661.—ABCD is a quadrilateral inscribed in a circle whose centre is O: if the angles BAD, BOD are supplementary, show that the arc BAD is double the arc BCD.

Ex. 662.—A, B, C are three points on a circle; D and E are mid-points of arcs AB, BC: the chords AE, CD cross at F. Show that EF = chord CE.

Ex. 663.—The join of the in- and circum-centres of a triangle subtend at each vertex an angle equal to half the difference of the angles at the other two vertices.

Ex. 664.—From any point P in the base BC of a triangle ABC, PE, PF are drawn parallel to two given straight lines to meet CA, AB in E, F. The circum-circle of PEF meets BC again in D. Show that the angles of DEF are the same for all positions of P.

Ex. 665.—F is any point in the side AB of a triangle ABC. Find D and E in BC, CA, so that the angles of triangle DEF shall be equal to those of a given triangle.

The last exercise should suggest the solution.

Ex. 666.—Let two circles touch externally at P, and let any straight line through P meet the circles again at A and B. Show how to draw another straight line through P meeting the circles in C and D, so that if tangents be drawn at A, B, O, D, they may enclose a rectangle.

Ex. 667.—D is any point in the base BC of an equilateral triangle ABC; circles ADCE, ADBF are drawn about triangles ADC, ADB; BF, CE are parallel to CA, AB. Show that the triangles ADE, ADF are equilateral. Hence describe an equilateral triangle with one vertex at a given point, and the others each on one of two given parallel straight lines.

Ex. 668.—If AB, AC are tangents at the points B, C of a circle, and if D is the mid-point of the arc BC, prove that D is the in-centre of triangle ABC.

Ex. 669.—The four straight lines bisecting the exterior angles of any quadrilateral form a cyclic quadrilateral.

Ex. 670.—A and B are two fixed points, and AC, AD are fixed straight lines, such that BA bisects angle CAD. If any circle passing through A

and B cut off the chords AK, AL from AC, AD, show that the sum of these chords is constant.

(*Use the corollaries to III.* 26, *III.* 29.)

Also if P be the other end of the diameter through A, show that the difference of the chords PK, PL is constant.

Ex. 671.—AB is a given chord of a circle APB ; P any point on the circle; the internal bisector of angle APB meets AB in Q. Show that the locus of the circum-centres of triangles APQ, BPQ consists of the four sides of a kite, of which AB is a diagonal.

Ex. 672.—PQ is the common chord of two circles which cut each other orthogonally: any point T is taken on either circle, and PT, QT, produced if necessary, cut the other circle in AB. Show that AB is a diameter of circle APQ. Also if AQ, BP meet in C, CT is the diameter of circle PTQ which is perpendicular to AB. Show also that the nine-point-circle of triangle ABC is fixed for all positions of T.

Ex. 673.—The ends of any diameter AT of a circle and the ends of the diameter BC perpendicular to AT of another circle, which cuts the first orthogonally, are also the ends of the diameters perpendicular to each other of two other pairs of circles which cut each other orthogonally.

Ex. 674.—A, B, C, D are concyclic. Show that the Simson lines of A, B, C, D, with respect to triangles BCD, CDA, DAB, ABC, and the nine-point-circles of those triangles, all pass through the same point.

Ex. 675.—A, B, C, D are concyclic. Show that the ortho-centres a, b, c, d of triangles BCD, CDA, DAB, ABC lie three by three on four equal circles. Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire.* (See Exs. 423-426, 533.)

Ex. 676.—A, B, C, D are concyclic. If a, b, c, d are the ortho-centres of triangles BCD, CDA, DAB, ABC, show that A, B, C, D are the ortho-centres of triangles bcd, cda, dab, abc.

Ex. 677.—Describe a square which shall have the ends of one diagonal on one given circle and the ends of the other diagonal on the other given circle. What condition must be satisfied that this may be possible?

THE HARPUR EUCLID

BOOKS V. and VI.

2 B

Books V. and VI. treat of Ratio and Proportion. Book V. explains the terms used and establishes general theorems; Book VI. applies these terms and theorems to plane figures.

On account of the difficult nature of Book V., it is usual to omit it from the student's course of reading; but he must acquaint himself with the terms used, and with many of the theorems established in it before he proceeds to Book VI. *These theorems he is allowed to assume as axioms;* but he will probably make much better progress in Book VI. if he sees how they follow from Euclid's definition of proportion (see Book V., Defs. 5 and 6). We shall therefore give proofs depending on the definitions. In these and in our illustrations of the definitions we shall use the notation recommended by De Morgan, and adopted by the Association for the Improvement of Geometrical Teaching in its *Syllabus* and *Elements.*

Our obligations to De Morgan's works are very great.

BOOK V.

DEFINITIONS.

1. A less magnitude is said to be a 'part' of a greater magnitude when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

The word '**part**' is used in the restricted sense of '**aliquot part**' in Arithmetic.

Sometimes a 'part' is called a 'measure.'

2. A greater magnitude is said to be a 'multiple' of a less when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

1 and 2 might be combined in one definition, thus :—

If one magnitude contains another a certain number of times exactly—

(1) **the less is called a 'part' or 'measure' of the greater;**

(2) **the greater is called a 'multiple' of the less.**

3. **'Ratio' is a mutual relation of two magnitudes of the same kind to one another in respect of quantity.**

We have given the definition in its usual form; but the word 'quantity' is misleading.

De Morgan translates thus :—

'Ratio' is a certain mutual habitude (σχέσις) **of two magnitudes of the same kind depending upon their quantuplicity** (πηλικότης).

We may put the matter thus : the magnitudes are not compared as to their 'how-much-ness' (*quantity*), but as to their 'how-manifold-ness' (*quantuplicity*).

Though the definition may not seem clear, Euclid's notion of ratio differs little from the common one. Thus the 'mutual relation' or 'mutual habitude' of a metre to a yard is roughly given by stating that 11 metres are about equal to 12 yards; or, with more accuracy, 32 metres are about equal to 35 yards. The 'quantuplicity' is estimated by a comparison of the multiples of a metre with those of a yard.

The ratio of A to B is denoted thus, A : B.

A is called the '**antecedent**' } of the ratio A : B.
B „ '**consequent**' }

4. **Magnitudes are said to have a 'ratio' to one another when the less can be multiplied so as to exceed the other.**

Here again we have followed the usual faulty translation. The definition should run thus:—

Magnitudes are said to have a ratio to one another when they can be multiplied so as to exceed the one the other.

[Rationem inter se habere magnitudines dicuntur quæ multiplicatæ altera alteram superare possunt.—(Heiberg).]

In its usual form the definition would seem merely to repeat the limitation of Def. 3, viz.:—to state, somewhat awkwardly, that the 'mutual relation' called 'ratio' only exists between magnitudes *of the same kind.* It seems probable, however, as De Morgan suggests, that Euclid intended it to point out that the two magnitudes must be such that, no matter how large a multiple of either of them was, a multiple of the other can be found which exceeds it. Thus, no matter how great a length a number of metres may make, a greater length can always be found containing an exact number of yards.

5. The first of four magnitudes is said to have the 'same ratio' to the second that the third has to the fourth, when, any equimultiples whatever of the first and third being taken, and any equimultiples whatever of the second and fourth, the multiple of the third is greater than, equal to, or less than that of the fourth, according as the multiple of the first is greater than, equal to, or less than that of the second.

This may be symbolically expressed thus: The ratio P : Q is said to be the same as the ratio X : Y, when, *m and n being any whole numbers whatever,*

$$mX \begin{matrix}>\\=\\<\end{matrix} nY \text{ according as } mP \begin{matrix}>\\=\\<\end{matrix} nQ.$$

Thus, to show that the ratio 4 : 7 is not the same as the ratio 9 : 16 according to Euclid's definition:—

Take the equimultiples 28 and 63 of 4 and 9, and the

equimultiples 28 and 64 of 7 and 16, and we see that while the multiples that we have taken of 4 and 7 are equal, those that we have taken of 9 and 16 are not equal.

The application of this test is easy when, as in the numerical example just given, a multiple of the first (7 times 4) can be found equal to a multiple (4 times 7) of the second, *i.e.* when the antecedent and consequent of the ratio considered have some common multiple (in our example 28). But the whole difficulty of the geometrical treatment of proportion springs from the fact that ratios have to be considered whose terms have no common multiple. Such quantities are said to be 'incommensurable,' since, as the student should easily be able to satisfy himself, they have no 'common measure' either. We shall afterwards show that the diagonal and the side of a square are incommensurable.

If it were not for the existence of such pairs of quantities, the ordinary Algebraical method by means of fractions would have been quite sufficient, and Euclid's more difficult treatment unnecessary.

6. Magnitudes which have the same ratio are called 'proportionals.'

Thus, if the ratio A : B is the same as the ratio of C : D, the four quantities A, B, C, D are called '**proportionals.**'

This relationship is denoted symbolically, thus:—

$$A : B :: C : D,$$

which may be read—

As A is to B so is C to D;
or, A is to B as C is to D.

Definitions 5 and 6 are of the greatest importance, as they contain Euclid's test and definition of proportion. The student must not be satisfied until he has thoroughly grasped Euclid's meaning when he asserts that four quantities are proportionals: probably he will not see its full force until after

he has mastered VI. 1, *which he is advised to read at once after learning Definition* 5.

Definition 5 may be stated thus:—

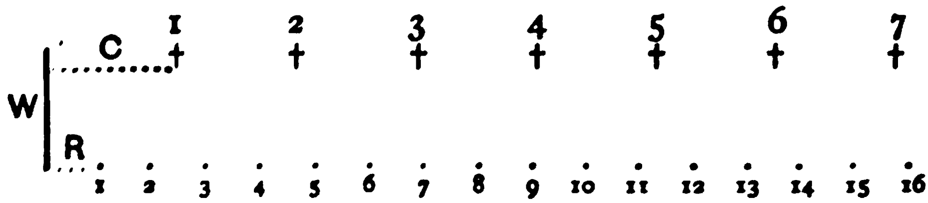
A is said to have to **B** the 'same ratio' that **X** has to **Y** when the multiples of **A** are distributed among those of **B** in the same manner as the multiples of **X** are distributed among those of **Y**;

Or, symbolically, thus—

$$A : B :: X : Y,$$

when, if mA lies between nB and $(n+1)$B,
mX lies between nY and $(n+1)$Y,
for all integral values of m and n.

Probably no better illustration of the force of Euclid's definition of proportion can be found than that which De Morgan uses in the article on 'Proportion' contributed by him to the *Penny Cyclopædia*, from which we proceed to quote in a condensed form:—



There is a straight colonnade composed of equidistant columns, the first being distant from a bounding-wall, **W**, by a length, **C**, equal to the distance between any two successive columns.

In front of the colonnade let there be a row of equidistant railings, the first being distant from **W** by a length, **R**, equal to the distance between any two successive railings, and let both columns and railings be numbered from the wall.

If we suppose this construction carried on to any extent, a spectator may without measurement compare the column-distance (**C**) with the railing-distance (**R**) to any degree of accuracy. For example, since the 10th railing falls between

the 4th and 5th columns, it follows that 10 R > 4 C and < 5C, or that R lies between $\frac{4}{10}$ C and $\frac{5}{10}$ C. To get a more accurate notion he may examine the 10,000th railing; if it fall between 4674th and 4675th columns, it follows that R lies between $\frac{4674}{10000}$ C and $\frac{4675}{10000}$ C. It can also be shown that the ratio of C to R is determined when the order of distribution of the railings among the columns is assigned *ad infinitum*. Any alteration, however small, in the place of the first railing must at last affect the order of distribution; suppose, for instance, that the first railing is moved further from the wall by ·001 of C; the second railing must then be pushed forward twice as much, the third three times as much, and so on: those after the 1000th are pushed forward by more than 1000 times as much, that is, by more than C; or the order with respect to the columns is disarranged.

Let it now be proposed to make a model of the preceding construction, in which c shall be the distance between the columns and r the distance between the railings. It needs no definition of proportion, nor anything more than the conception which we have of that term prior to definition, to assure us that C must be to R in the same proportion as c to r if the model be truly formed. Nor is it drawing too largely on that conception of proportion if we assert that the distribution of the railings among the columns in the model must be everywhere the same as in the original; for example, that the model would be out of proportion if its 56th railing fell between the 18th and 19th column while the 56th railing of original fell between the 17th and 18th columns. The obvious relation between the construction and its model contains the collection of conditions, the fulfilment of which, according to Euclid, constitutes proportion.

7. When of the equimultiples of four magnitudes (taken as in the 5th definition) the multiple of the first is greater than that of the second, but the

multiple of the third is not greater than that of the fourth, then the first is said to have to the second a 'greater ratio' than the third has to the fourth.

This may be expressed symbolically thus :—

If A, B, X, Y are four magnitudes and two whole numbers, m and n can be found such that

$$mA > nB, \text{ while at the same time } mX \overset{=}{<} nY;$$

the ratio A : B is said to be greater than the ratio X : Y.

8. 'Analogy' or 'proportion' is the similitude of ratios.

9. Proportion consists in three terms at least.

10. When three magnitudes are proportionals the first is said to have to the third the 'duplicate ratio' of that which it has to the second.

Thus if A : B :: B : C,
then the ratio of A : C is called the '**duplicate**' of the ratio A : B.

11. When four magnitudes are continual proportionals the first is said to have to the fourth the 'triplicate ratio' of that which it has to the second, and so on, 'quadruplicate,' etc., increasing the denomination still by unity in any number of proportionals.

Thus if A : B :: B : C,
and B : C :: C : D
C : D :: D : E,

the ratio A : D is called the '**triplicate**' } of the
and „ A : E „ '**quadruplicate**' } ratio A : B.

DEFINITION OF COMPOUND RATIO.

When there are any number of magnitudes of the same kind, the first is said to have to the last of

them the ratio 'compounded of' the ratio which the first has to the second, of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude.

Thus if A, B, C, D be four magnitudes **of the same kind**, the ratio A : D is said to be '**compounded of**' the ratios A : B, B : C, C : D.

$$\text{Also if} \begin{cases} A : B :: E : F, \\ B : C :: G : H, \\ C : D :: K : L, \end{cases}$$

then the ratio of A : D is said to be '**compounded of**' the ratios of E : F, G : H, K : L;

And further, if, *the same things being supposed*, M : N :: A : D, then the ratio M : N is also said to be '**compounded of**' the ratios E : F, G : H, K : L.

There are few terms about which a student's notions are apt to be more hazy than about 'compound ratio.' Special attention must be given to it and to the propositions of Book VI. in which it occurs.

Note that *if the ratios* A : B, B : C *are equal*, the ratio 'compounded of' them is called the '**duplicate**' of the ratio A : B. (See Def. 10.)

Similarly, if the ratios A : B, B : C, C : D *are all equal*, the ratio 'compounded of' them is called the '**triplicate**' of the ratio A : B, and so on. (See Def. 11.)

12. In proportionals, the antecedent terms are called 'homologous' to one another; so also are the consequents.

Thus if A : B :: C : D,
the terms A and C are said to be '**homologous**' to one another; so also are the terms B and D.

Geometers make use of the following technical words (*permutando, invertendo, componendo, dividendo, convertendo, ex æquali*) to signify certain ways of **changing either the**

order or the magnitude of proportionals so that they still **continue to be proportionals.**

13. 'Permutando' or 'alternando,' by permutation or alternately; used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth.

It is demonstrated in V. 16 that if A, B, C, D be four quantities **of the same kind** such that A : B :: C : D,

then A : C :: B : D.

14. 'Invertendo,' by inversion; used when there are four proportionals, and it is inferred that the second is to the first as the fourth is to the third.

It is demonstrated in 'V. B' that if A : B :: C : D,

then B : A :: D : C.

The ratio B : A is called the '**reciprocal**' of the ratio A : B.

15. 'Componendo,' by composition; used when there are four proportionals, and it is inferred that the first together with the second is to the second as the third together with the fourth is to the fourth.

It is demonstrated in V. 18 that if A : B :: C : D,

then A+B : B :: C+D : D.

16. 'Dividendo' by division; used when there are four proportionals, and it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.

It is demonstrated in V. 17 that if A+B : B :: C+D : D,

then A : B :: C : D.

17. 'Convertendo,' by conversion; used when there are four proportionals, and it is inferred that the first is to its excess above the second as the third is to its excess above the fourth.

It is demonstrated in 'V. E' that if $A+B : B :: C+D : D$,
then $A+B : A :: C+D : C$.

18. **'Ex æquali,' or 'ex æquo,' from equality of distance; used when there is any number of magnitudes more than two and as many others so that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes as the first is to the last of the others.**

Of this there are the following two kinds which arise from *the different order in which the magnitudes are taken, two and two*:—

19. It is demonstrated in V. 22 that—

If A, B, C are three magnitudes of the same kind, and X, Y, Z form another set of magnitudes, such that

$$A : B :: X : Y,$$
$$\textbf{and } B : C :: Y : Z,$$
$$\textbf{then } A : C :: X : Z.$$

The demonstration can be easily extended to any number of magnitudes.

20. It is demonstrated in V. 23 that:—

If A, B, C are three magnitudes of the same kind, and X, Y, Z form another set of three, such that

$$A : B :: Y : Z$$
$$\textbf{and } B : C :: X : Y,$$
$$\textbf{then } A : C :: X : Z.$$

The demonstration can be easily extended to any number of magnitudes.

From 19 and 20 we see that—

(**'Ratios compounded of equal ratios are equal.'** See definition of **'Compound Ratio.'**)

AXIOMS.

1. Equimultiples of the same or of equal magnitudes are equal to one another.

2. Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another is greater than that other magnitude.

In what follows we adopt the notation of the *Syllabus*.

Large Roman letters, A, B, etc., are used to denote magnitudes, and where the pairs of magnitudes are both of the same kind they are denoted by letters taken from the same part of the alphabet, as A, B compared with C, D; but where they are or may be of different kinds, from different parts of the alphabet, as A, B compared with P, Q or X, Y.

Small Italic letters, *m*, *n*, etc., denote whole numbers.

By *m*A is denoted the *m*th multiple of A, and it may be read as '*m* times A.'

PROPOSITION 1.

Let there be any number of magnitudes A, B, C ... which are equimultiples respectively of A', B', C'; then the sum of A, B, C ... is the same multiple of the sum of A', B', C' as A is of A', B of B', etc.

$$\text{Suppose } A=2A',\ B=2B',\ C=2C', \text{ etc.,}$$

$$\begin{aligned} \text{then } A+B &= A'+A'+B'+B', \\ &= A'+B'+A'+B', \\ &= 2(A'+B'). \end{aligned}$$

Hence $A+B+C=2(A'+B'+C')$, and so on.

Similarly if A, B, C be treble, quadruple, etc., of A', B', C' respectively.

NOTE.

We learn from this theorem that

$$mA+mB+mC...=m(A+B+C...).$$

The particular case demonstrated has been assumed in III. 21.

The proposition asserts that

$$mA+mB+mC...=m(A+B+C...)$$

for any number of magnitudes A, B, C, and any value of the positive integer m.

PROPOSITION 2.

If A and C are equimultiples of B and D, and E and F are also equimultiples of B and D, then A+E and C+F are equimultiples of B and D.

$$\text{For let } A=mB \text{ and } E=nB,$$

$$\text{then } C=mD \text{ and } F=nD,$$

$$\left.\begin{aligned} \text{then } A+E &= mB+nB, \\ &= (m+n)B, \\ \text{and } C+F &= (m+n)D. \end{aligned}\right\}$$

PROPOSITION 3.

If A and C are equimultiples of B and D, and E and F are equimultiples of A and C, then E and F are equimultiples of B and D.

$$\begin{aligned} \text{Let } A &= 2B, \\ \text{and } C &= 2D, \\ E &= 3A, \\ \text{and } F &= 3C. \\ \text{Then } E &= A+A+A, \\ &= 2B+2B+2B. \\ \text{Similarly } F &= 2D+2D+2D; \end{aligned}$$

$\therefore$ E and F are equimultiples of B and D.

Similarly for other equimultiples, and generally

$$q(mB) \text{ and } q(mD)$$

are always equimultiples of B and D.

PROPOSITION 4

$$\text{If } A : B :: C : D,$$

$$\text{then } mA : nB :: mC : nD.$$

For $p(mA)$, $p(mC)$ are equimultiples of A and C, and $q(nB)$, $q(nD)$ are equimultiples of B and D;

$$\therefore p(mC) \gtreqless q(nD) \text{ according as } p(mA) \gtreqless q(nB);$$

$$\therefore mA : nB :: mC : nD.$$

PROPOSITION 5.

$$\begin{aligned} \text{If } A+B &= m(A'+B'), \\ \text{and } B &= mB', \\ \text{then } A &= mA'. \end{aligned}$$

$$\begin{aligned} \text{For } A+B &= m(A'+B'), \\ &= mA'+mB', \\ \text{and } B &= mB'; \\ \therefore\ A &= mA'. \end{aligned}$$

Hence if $P > Q$

$$mP - mQ = m(P-Q).$$

PROPOSITION 6.

If $A+E$ and $B+F$ be equimultiples of C and D, and E and F be equimultiples of C and D, then A and B are either equimultiples of C and D or equal to them.

$$\begin{aligned} \text{Let } A+E &= (m+n)C, \\ B+F &= (m+n)D, \\ E &= nC, \\ F &= nD. \end{aligned}$$

$$\begin{aligned} \text{Then } A+E &= (m+n)C, \\ &= mC+nC, \\ \text{and } E &= nC; \\ \therefore\ A &= mC. \end{aligned}$$

Similarly $B = mD$.

Hence if $m > n$, $(m-n)A = mA - nA$.

PROPOSITION A.

If $A : B :: P : Q$,

then $P \gtreqless Q$ according as $A \gtreqless B$.

For $2P \gtreqless 2Q$ according as $2A \gtreqless 2B$; [V. Def. 5.

$\therefore$ $P \gtreqless Q$ according as $A \gtreqless B$.

PROPOSITION B. (Invertendo.)

If A : B :: P : Q,
then B : A :: Q : P.

Take any equimultiples mA, mP of A and P, and any equimultiples nB, nQ of B and Q.

Then if $nB > mA$,
$mA < nB$;
and $\therefore$ $mP < nQ$, [V. Def. 5.
i.e. $nQ > mP$.

Similarly if $nB \underset{<}{=} mA$,

$nQ \underset{<}{=} mP$;

$$\therefore\ nQ \overset{>}{\underset{<}{=}} mP \text{ according as } nB \overset{>}{\underset{<}{=}} mA;$$

$\therefore$ B : A :: Q : P.

Or thus :—

If the multiples of A are distributed among those of B in the same way as the multiples of P are distributed among those of Q, it follows that the multiples of B are distributed among those of A in the same way as the multiples of Q are distributed among those of P.

Hence :—**Reciprocals of equal ratios are equal.**

PROPOSITION C.

$$\text{If}\begin{cases} A = mB \\ \text{and } P = mQ \end{cases} \text{or if}\begin{cases} mA = B \\ \text{and } mP = Q, \end{cases}$$

then A : B :: P : Q.

We leave these as an exercise to the student on V. Def. 5.

PROPOSITION D.

If $A : B :: P : Q$,
and $A = mB$,
then $P = mQ$.
Also if $A : B :: P : Q$,
and $mA = B$,
then $mP = Q$.

We leave these as an exercise to the student.

PROPOSITION 7.

If $A = B$, then (1) $A : C :: B : C$;
(2) $C : A :: C : B$.

Take any equimultiples mA, mB of A and B, and any multiple nC of C.

Then $mA = mB$.

And if $mA > nC$,
then $mB > nC$.

Similarly if $mA \overset{=}{<} nC$,

then $mB \overset{=}{<} nC$;

$$\therefore mB \begin{matrix} > \\ = \\ < \end{matrix} nC \text{ according as } mA \begin{matrix} > \\ = \\ < \end{matrix} nC;$$

$\therefore A : C :: B : C$;
$\therefore$ also $C : A :: C : B$. [INVERTENDO V.B.

Or thus :—

If two magnitudes are equal, their multiples must be distributed in the same way among the multiples of any third magnitude; and the multiples of any third magnitude must be distributed in the same way among the multiples of one of the equal magnitudes as among those of the other.

Hence generally: **Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.**

PROPOSITION 8.

(1) If $A > B$,
then $A : C > B : C$.
Let $A = B + K$,
and let mK be any multiple of K greater than C.
Suppose mA lies between nC and $(n+1)$ C,
then $mB + mK$ lies between nC and $\overline{n+1}$ C.
But $mK > C$;
$\therefore mB < nC$;
$\therefore A : C > B : C$. [V. DEF. 7.
(2) If $A > B$,
$C : B > C : A$.
As in (1) $nC > mB$, while $nC < mA$.

PROPOSITION 9.

(1) If $A : C :: B : C$,
then $A = B$.

For if $A > B$, it may be demonstrated as in V. 8 (1), that there are multiples such that $mA > nC$, while $mB < nC$, which is contrary to the hypothesis.

Similarly A cannot be less than B.

(2) If $C : A :: C : B$,
then $A = B$.

For if $A > B$, it can be demonstrated, as in V. 8, that there are multiples such that

$nC > mB$, while $nC < mA$,

which is contrary to the hypothesis.

Similarly A cannot be less than B.

Hence :—

(1) **Magnitudes which have the same ratio to the same magnitude are equal to one another.**

(2) **Magnitudes to which the same magnitude has the same ratio are equal to one another.**

PROPOSITION 10.

(1) If $A : C > B : C$,
then $A > B$.
$\because A : C > B : C$,
there exist multiples mA, mB, nC,
such that $mA > nC$, while $mB \overset{=}{<} nC$; [V. DEF. 7.
$\therefore mA > mB$;
$\therefore A > B$.

(2) If $C : B > C : A$,
then $B < A$.
$\because C : B > C : A$,
there exist multiples mA, mB, nC,
such that $nC > mB$, while $nC \overset{=}{<} mA$; [V. DEF. 7.
$\therefore mB < mA$;
$\therefore B < A$.

PROPOSITION 11.

If $A : B :: P : Q$,
and $P : Q :: X : Y$,
then $A : B :: X : Y$.

Take any equimultiples mA, mP, mX of A, P and X, and any equimultiples nB, nQ, nY of B, Q, and Y.

Now if $mA > nB$,

$mP > nQ$ ($\because$ A : B :: P : Q); [V. Def. 5.

$\therefore$ $mX > nY$ ($\because$ P : Q :: X : Y). [V. Def. 5.

Similarly it may be shewn that

$$mX \gtreqless nY \text{ according as } mA \gtreqless nB;$$

$\therefore$ A : B :: X : Y. [V. Def. 5.

Hence generally :—

Ratios that are the same to the same ratio are the same to one another.

PROPOSITION 12.

If A : B :: C : D :: E : F,
then A+C+E : B+D+F :: A : B.

Take any equimultiples mA, mC, mE of A, C, E,
and any equimultiples nB, nD, nF of B, D, F.

If $mA > nB$,
then $mC > nD$,
and $mE > nF$;

$\therefore$ $mA + mC + mE > nB + nD + nF$;

$\therefore$ $m(A+C+E) > n(B+D+F)$. [V. 1.

Similarly, $m(A+C+E) \gtreqless n(B+D+F)$ according as $mA \gtreqless nB$;

$\therefore$ A+C+E : B+D+F :: A : B.

PROPOSITION 13.

If A : B :: P : Q,
and P : Q > X : Y,
then A : B > X : Y.

$\cdot$ P : Q > X : Y;

$\therefore$ multiples $m\mathsf{P}$, $n\mathsf{Q}$, $m\mathsf{X}$, $n\mathsf{Y}$ can be found

such that $m\mathsf{P} > n\mathsf{Q}$ while $m\mathsf{X} \overset{=}{<} n\mathsf{Y}$.

But if $m\mathsf{P} > n\mathsf{Q}$,

then $m\mathsf{A} > n\mathsf{B}$;

$\therefore \mathsf{A} : \mathsf{B} > \mathsf{X} : \mathsf{Y}$.

PROPOSITION 14.

If $\mathsf{A} : \mathsf{B} :: \mathsf{C} : \mathsf{D}$,

then $\mathsf{B} \gtreqless \mathsf{D}$ according as $\mathsf{A} \gtreqless \mathsf{C}$.

Let $\mathsf{A} > \mathsf{C}$,

then ratio $\mathsf{A} : \mathsf{B} >$ ratio $\mathsf{C} : \mathsf{B}$; [V. 8.

$\therefore$ also ratio $\mathsf{C} : \mathsf{D} >$ ratio $\mathsf{C} : \mathsf{B}$; [V. 13.

$\therefore \mathsf{B} > \mathsf{D}$.

Similarly if $\mathsf{A} < \mathsf{C}$,

then $\mathsf{B} < \mathsf{D}$.

Also if $\mathsf{A} = \mathsf{C}$, then $\mathsf{B} = \mathsf{D}$. [V. 9.

PROPOSITION 15.

If $m\mathsf{A}$, $m\mathsf{B}$ are any equimultiples of A and B,

then $m\mathsf{A} : m\mathsf{B} :: \mathsf{A} : \mathsf{B}$.

If $\mathsf{A} = \mathsf{C} = \mathsf{E}$,

and $\mathsf{B} = \mathsf{D} = \mathsf{F}$,

then $\mathsf{A} : \mathsf{B} :: \mathsf{C} : \mathsf{D} :: \mathsf{E} : \mathsf{F}$; [V. 7.

$\therefore \mathsf{A}+\mathsf{C}+\mathsf{E} : \mathsf{B}+\mathsf{D}+\mathsf{F} :: \mathsf{A} : \mathsf{B}$, [V. 12.

i.e. $3\mathsf{A} : 3\mathsf{B} :: \mathsf{A} : \mathsf{B}$,

and similarly for any other equimultiples.

Hence generally :—

Magnitudes have the same ratio to one another that their equimultiples have.

PROPOSITION 16. (PERMUTANDO.)

If A, B, C, D be four magnitudes of the same kind such that

$$A : B :: C : D,$$

then $A : C :: B : D$.

Take any equimultiples mA, mB of A and B, and any equimultiples nC, nD of C and D;

then $mA : mB :: A : B$, [V. 15.

$:: C : D$. [Hyp.

$:: nC : nD$. [V. 15.

$\therefore mB \gtreqless nD$ according as $mA \gtreqless nC$; [V. 14.

$\therefore A : C :: B : D$. [V. Def. 5

PROPOSITION 17. (DIVIDENDO.)

If $A+B : B :: C+D : D$,

then $A : B :: C : D$.

Take any equimultiples mA, mB, mC, mD of A, B, C, D, and also any equimultiples nB, nD of B and D.

$$\text{If } mA > nB,$$
$$mA+mB > nB+mB;$$
$$\therefore m(A+B) > (m+n)B;$$
$$\therefore m(C+D) > (m+n)D;$$
$$\therefore mC+mD > mD+nD;$$
$$\therefore mC > nD.$$

Similarly it can be shown that

$$mC \leqq nD \text{ according as } mA \leqq nB;$$

$$\therefore A : B :: C : D.$$

PROPOSITION 18. (Componendo.)

If $A : B :: C : D$,
then $A+B : B :: C+D : D$.

Take any equimultiples $m(A+B)$, mB, $m(C+D)$, mD of
$A+B$, B, $C+D$, D,
and also any equimultiples nB, nD of B and D.

If $m(A+B)>nB$,
$mA+mB>nB$;
$\therefore mA>nB-mB$;
$\therefore mA>(n-m)B$; [V. 6.
$\therefore mC>(n-m)D$; [V. Def. 5.
$\therefore mC>nD-mD$; [V. 5.
$\therefore mC+mD>nD$;
$\therefore m(C+D)>nD$.

Similarly, $m(C+D)\gtreqless nD$ according as $m(A+B)\gtreqless nB$;

$\therefore A+B : B :: C+D : D$.

The proof supposes $m<n$,
if $m>n$, it is obvious that $m(A+B)>nB$,
and $m(C+D)>nD$.

PROPOSITION 19.

If $A+E : B+F :: E : F$,
then $A : B :: A+E : B+F$.

$\because A+E : B+F :: E : F$;
$\therefore A+E : E :: B+F : F$; [V. 16, Alt.
$\therefore A : E :: B : F$; [V. 17, Div.
$\therefore A : B :: E : F$.
$:: A+E : B+F$. [Hyp.

PROPOSITION E. (CONVERTENDO.)

If $A+B : B :: C+D : D$,
then $A+B : A :: C+D : C$.
$\because A+B : B :: C+D : D$;
$\therefore A : B :: C : D$; [V. 17.
$\therefore B : A :: D : C$; [V. B. INVERTENDO
$\therefore A+B : A :: C+D : C$. [V. 18. COMP

PROPOSITION 20.

If $A : B :: P : Q$,
and $B : C :: Q : R$,

then $P \underset{<}{\overset{>}{=}} R$ according as $A \underset{<}{\overset{>}{=}} C$.

If $A > C$,
then $A : B > C : B$; [V. 8.
and $\therefore P : Q > R : Q$; [V. B. and V. 13.
$\therefore P > R$.

Similarly, $P \underset{<}{=} R$ according as $A \underset{<}{=} C$.

PROPOSITION 21.

If $A : B :: Q : R$,
and $B : C :: P : Q$;

then $P \underset{<}{\overset{>}{=}} R$ according as $A \underset{<}{\overset{>}{=}} C$.

For if $A > C$,
$A : B > C : B$;

and $\therefore$ Q : R > Q : P; [V. B. and V. 13.

$\therefore$ R < P,

i.e. P > R.

Similarly it can be shewn that

$$P \substack{>\\=\\<} R \text{ according as } A \substack{>\\=\\<} C.$$

PROPOSITION 22. Ex Æquali.

If A : B :: P : Q,

and B : C :: Q : R;

then A : C :: P : R.

For $mA : nB :: mP : nQ$, and $nB : qC :: nQ : qR$; [V. 4.

$\therefore$ $mP \substack{>\\=\\<} qR$ according as $mA \substack{>\\=\\<} qC$; [V. 20.

$\therefore$ A : C :: P : R.

Hence also if A : B :: P : Q,

B : C :: Q : R,

C : D :: R : S,

then A : D :: P : S.

For A : C :: P : R, [By 1st Part.

and C : D :: R : S; [Hyp.

$\therefore$ A : D :: P : S, [By 1st Part.

and the theorem can easily be extended to any number of magnitudes.

Hence generally :—

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

PROPOSITION 23. Ex æquo perturbato.

If $A : B :: Q : R$,
and $B : C :: P : Q$,
then $A : C :: P : R$.

For $mA : mB :: nQ : nR$, [V. 15.
and $mB : nC :: mP : nQ$; [V. 4.

$\therefore$ $mP \gtreqless nR$ according as $mA \gtreqless nC$; [V. 21.

$\therefore$ $A : C :: P : R$.

Hence also if $A : B :: R : S$,
$B : C :: Q : R$,
$C : D :: P : Q$,
then $A : D :: P : S$.

For $\because$ $A : B :: R : S$,
and $B : C :: Q : R$;
$\therefore$ $A : C :: Q : S$; [1st Part.
but $C : D :: P : Q$; [Hyp.
$\therefore$ $A : D :: P : S$, [1st Part.

and the theorem can be easily extended to any number of magnitudes.

Hence generally :—

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

PROPOSITION 24. Addendo.

If $A : B :: P : Q$,
and $C : B :: R : Q$,
then $A+C : B :: P+R : Q$.

For $B : C :: Q : R$, [Invertendo.
and $A : B :: P : Q$; [Hyp.
$\therefore A : C :: P : R$; [Ex æquali.
$\therefore A+C : C :: P+R : R$. [Componendo.
But $C : B :: R : Q$; [Hyp.
$\therefore A+C : B :: P+R : Q$. [Ex æquali.

PROPOSITION 25.

If A, B, C, D be four magnitudes of the same kind such that $A : B :: C : D$,
and if A be the greatest and consequently D the least of them, then $A+D > B+C$.

$\because A : B :: C : D$,
$\therefore A-C : B-D :: A : B$. [V. 19.
But $A > B$,
$\therefore A-C > B-D$.
Add $C+D$ to each,
then $A+D > B+C$.

BOOK VI.

Def.—The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

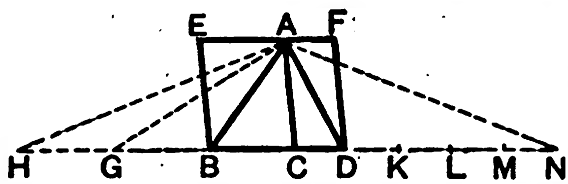
PROPOSITION 1. Theorem.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let the $\triangle$s ABC, ACD and the ||gms EC, CF have the same altitude, viz. the ⊥r from A to BD; then

(1) BC : CD :: $\triangle$ABC :: $\triangle$ACD;

(2) BC : CD :: ||gm EC : ||gm CF.



Produce BD both ways, and cut off *any number* of st. lines BG, GH, each equal to BC, and *any number* DK, KL, LM, MN, each equal to CD.

Join AG, AH, AN.

∵ BC = BG = GH,

∴ $\triangle$ABC = $\triangle$ABG = AGH;

∴ CH and $\triangle$ACH are equimultiples of BC and $\triangle$ABC.

Similarly CN and $\triangle$ACN are equimultiples of CD and $\triangle$ACD.

Now if CH = CN,

$\triangle$ACH = $\triangle$ACN, [I. 38.

and it easily follows that if CH > CN,

$\triangle$ACH > ACN,

and if CH < CN,

$\triangle$ACH < $\triangle$ACN;

∴ BC : CD :: $\triangle$ABC : $\triangle$ACD.

[For we have four magnitudes BC, CD, $\triangle$ABC, $\triangle$ACD,

and of the first and third any equimults. CH and $\triangle$ACH, and of the second and fourth any equimults. CN and $\triangle$ACN have been taken, and it has been found that $\triangle ACH \gtreqless \triangle ACN$ according as $CH \gtreqless CN$.]

Again ||gm EC : ||gm CF :: $\triangle$ABC : $\triangle$ACD. [V. 11.
:: BC : CD.

COR.—From this it is plain that triangles and parallelograms that have equal altitudes are to one another as their bases.

[Place them so that their bases are in the same straight line and the figures on the same side of it; then they will be between the same parallels as in the demonstration.]

NOTE.

The 'two triangles of the same altitude' need not have a common side as in the figure given.

The student should vary the figure, giving them for instance a common vertex, but not a common side, or taking any two triangles between the same parallels as indicated in the Corollary. Observe carefully the force of the expression '*any number* of st. lines.'

Ex. 678.—Prove VI. 1 by demonstrating first for the parallelograms and then deducing for the triangles.

Ex. 679.—Triangles on the same base are to one another as their altitudes. Hence show also that triangles on equal bases are to one another as their altitudes.

Ex. 680.—Show by Ex. 679 and V. 24 that the sum of the perpendiculars to the equal sides of an isosceles triangle from any point in the base is constant (see Ex. 133).

Show also that the sum of the perpendiculars from any point within an equilateral triangle to the three sides is constant.

Ex. 681.—ABC, DBC are two triangles on the same base BC: the line joining the vertices A and D cuts the base in E.

Show that $\triangle$ABD : $\triangle$ADC :: BE : EC.
Hence find a point O within $\triangle$ABC,
such that $\triangle AOB = \triangle BOC = \triangle COA$.

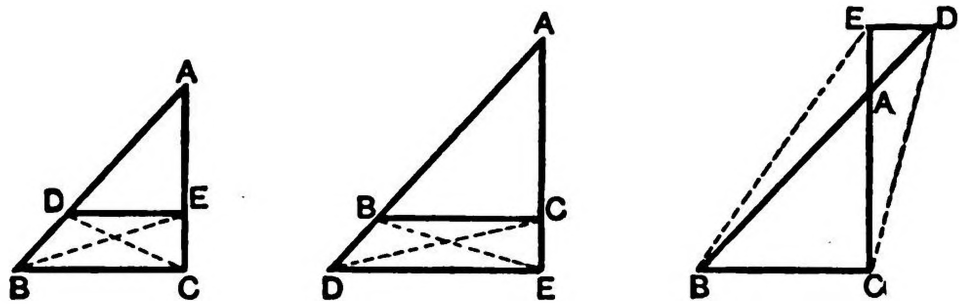
Ex. 682.—Find a pt. O within or without a given $\triangle$ABC, such that $\triangle$AOB : $\triangle$BOC : $\triangle$COA :: AB : BC : CA.

PROPOSITION 2. THEOREM.

(1) **If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those sides produced proportionally.**

(2) Conversely :—**If two sides of a triangle, or those sides produced, be cut proportionally, the straight line joining the points of section shall be parallel to the base.**

(1) Let the st. line DE, ∥ to the sides BC of a △ABC, cut AB, AC, or AB, AC produced in D, E;
then BD : DA :: CE : EA.



Join BE, CD.
△BED=△CED (on same base and between same ∥s),
and ADE is another △;
∴ △BDE : △ADE :: △CDE : △ADE.
But △BDE : △ADE :: BD : DA,
and △CDE : △ADE :: CE : EA; [VI. 1.
∴ BD : DA :: CE : EA.

(2) Next let DE cut AB, AC or AB, AC produced,
so that BD : DA :: CE : EA,
then DE is ∥ to BC.

Now BD : DA :: △BDE : △ADE,
CE : EA :: △CDE : △ADE, [VI. 1.
but BD : DA :: CE : EA;
∴ △BDE : △ADE :: △CDE : △ADE;

$\therefore \triangle BDE = \triangle CDE$;
$\therefore$ DE is || to BC.

Note that D is on AB or AB produced *in all 3 diagrams*. Students frequently interchange D and E in the third diagram, thereby rendering their demonstration inapplicable to it.

Ex. 683.—Show that in each of the figures for VI. 2
(1) if DE is || to BC,
then AB : AD :: AC : AE.
(2) if AB : AD :: AC : AE,
then DE is || to BC.
Using VI. 2 and *componendo* or *dividendo*.
Deduce the same results by VI. 1 independently of VI. 2.

Ex. 684.—If CD, BE cut in F, show that AF is a median of $\triangle$ ABC. (Diag. of VI. 2. Use Ex. 681).

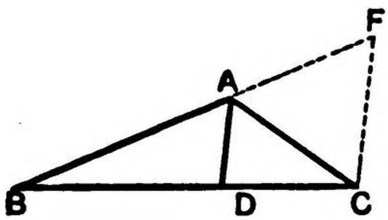
Ex. 685.—Enunciate and prove the converse of the last exercise.

PROPOSITION 3. THEOREM.

(1) **If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another.**

(2) Conversely:—**If the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.**

(1) Let the st. line AD bisect the vertl. ∠ BAC of the △ABC, and meet BC in D; then BD : DC :: BA : AC.



Through C draw CF ‖ to AD,
meeting BA produced in F.
Then ∠ ACF = alt. ∠ CAD. [I. 29.
= ∠ BAD, [Hyp.
= int. ∠ AFC; [I. 29.
∴ AF = AC.
But BD : DC :: BA : AF (∵ CF is ‖ to AD);
∴ BD : DC :: BA : AC. [V. 7.

(2) Next let AD cut BC, so that
BD : DC :: BA : AC;
then AD shall bisect ∠ BAC.
With the same construction as in (1)
BA : AF :: BD : DC (∵ CF is ‖ to AD),
:: BA : AC; [Hyp.

$\therefore$ AF = AC;
$\therefore$ $\angle$ AFC = $\angle$ ACF
But $\angle$ BAD = int. $\angle$ AFC, [I. 29.
= $\angle$ ACF,
= alt. $\angle$ CAD;
$\therefore$ AD bisects $\angle$ BAC.

Ex. 686.—If in the fig. of VI. 3 we draw DH, CK $\perp$r to AD to meet AB in H, K, then

AB : AK :: HB : KH.

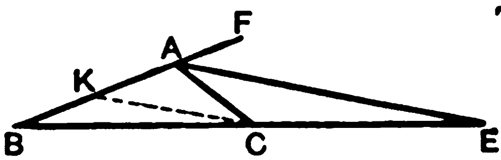
A useful trigonometrical expression for AD *can be deduced from this proportion.*

PROPOSITION A. THEOREM.

(1) **If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line, which also cuts the base produced, the segments between the dividing straight line and the extremities of the base shall have the same ratio which the other sides of the triangle have to one another.**

(2) Conversely:—**If the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section shall bisect the exterior angle of the triangle.**

(1) Let the st. line AE bisect the exterior ∠ CAF of △BAC and meet BC produced in E; then BE : EC :: BA : AC.
Through C draw CK ∥ to AE, meeting BA in K.



Then ∠ ACK = alt. ∠ CAE,
= ∠ FAE, [Hyp.
= int. ∠ AKC;
∴ AK = AC.

But BE : EC :: BA : AK (∵ AE is ∥ to KC);
∴ BE : EC :: BA : AC. [V. 7.

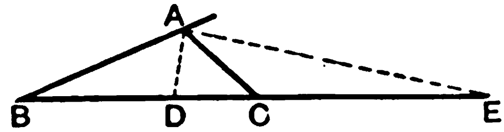
(2) Next let AE cut BC produced in E,
so that BE : EC :: BA : AC,
then AE shall bisect ext. ∠ CAF.

With the same construction
BA : AK :: BE : EC (∵ AE is ∥ to KC),
:: BA : AC; [Hyp.
∴ AK = AC; [V. 7.
∴ ∠ AKC = ∠ ACK.
But ∠ FAE = int. ∠ AKC,
= ∠ ACK,
= alt. ∠ CAE;
∴ AE bisects ∠ CAF.

The student should note that—

If the internal and external bisectors of the vertical angle A cut the base BC of a triangle ABC in D, E, then

BE : EC :: BD : DC.
(For each ratio = BA : AC).



The line BC thus divided *internally and externally in the same ratio* is said to be **harmonically divided.**

It may interest the student to note that in such a case, by *alternando*

BE : BD :: EC : DC,

i.e. BE : BD :: BE − BC :: BC − BD,

the lines BE, BC, BD thus satisfying the Algebraical definition of Harmonic Progression.

Hence BC is called the *Harmonic Mean* of BD and BE.

The consideration of harmonically divided lines forms an important branch of Modern Geometry. Hence a short treatise on **Harmonic Division** will be given among the Addenda to Book VI.

Ex. 687.—Show that ED is the harmonic mean between EB and EC.

Ex. 688.—If DK be drawn ⊥r to AD to meet AB or AC in K, then AK is the harmonic mean of AB, AC (see Ex. 686).

Show also that if AB and AC are unequal, AK is less than half their sum.

Draw a parallel to DK *through the mid point of* BC.

To what proposition in Algebra does this correspond?

Ex. 689.—If through a fixed point D on the internal bisector of an angle BAC *any* straight line BDC be drawn, cutting AB, AC in B, C, then the harmonic mean of AB, AC is constant.

Ex. 690.—State and prove the converse of the last Ex.

Ex. 691.—A straight line DEF is drawn, cutting the sides BC, CA, AB of a △ABC, or those sides produced in D, E, F, and equally inclined to AB, AC; then BD : DC :: BF : EC.

Prove this (1) independently of VI. 3 and VI. A. by means of a parallel to DEF through B or C;

(2) by means of VI. 3 and VI. A.

Our attention was drawn to this generalisation of VI. 3 and VI. A. by Mr. R. Levett, M.A.

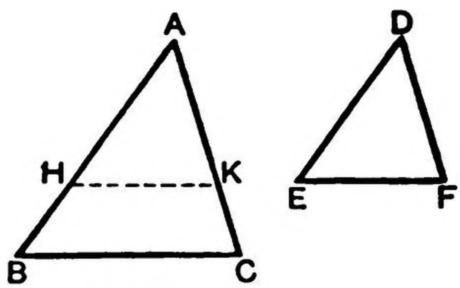
DEF.—**Similar rectilineal figures are those which have their several angles equal each to each and the sides about the equal angles proportional.**

PROPOSITION 4. THEOREM.

The sides about the equal angles of triangles which are equiangular to one another are proportionals, those sides being homologous which are opposite to equal angles.

Let $\angle$s A, B, C of $\triangle$ABC $=$ $\angle$s D, E, F of $\triangle$DEF respectively,

then BA : AC :: ED : DF,
AC : CB :: DF : FE;
and $\therefore$ BA : CB :: ED : FE. [EX ÆQUALI.



From AB, AC, produced if necessary, cut off AH, AK equal to DE, DF, and join HK.

Then $\angle$AHK $=$ $\angle$E, [I. 4.
$=$ $\angle$B; [HYP.
$\therefore$ HK is $\parallel$ to BC;
$\therefore$ BH : HA :: CK : KA; [VI. 2.
$\therefore$ AB : AH :: AC : AK; [COMPONENDO.
$\therefore$ AB : AC :: AH : AK, [ALTERNANDO.
:: DE : DF.
Similarly AC : CB :: DF : FE,
and CB : BA :: FE : ED.

Cor.—**Hence if two triangles are equiangular to one another, they are similar.**

N.B.—The student may easily see by the simple case of a square and oblong that the same proposition does not hold true for quadrilaterals.

Def.—**If the angles of a rectilineal figure are given, and the ratios of its sides also given, it is said to be given in species.**

From VI. 4 we see that—

If a triangle is equiangular to a given triangle, it is 'given in species';

and hence also that—

If each of the angles of a triangle be given in magnitude, the triangle is 'given in species' (Euclid's *Data*, 43).

Ex. 692.—If, in the fig. of VI. 3, ∠ BAC is double of ∠ B, AC is a mean propl. between BC and CD.

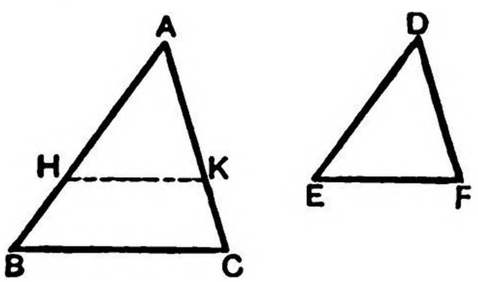
Ex. 693.—The chds. AB, DC of a ⊙ cross at E. Show that △s AED, BEC are similar. If AB, CD be produced to meet at F, shew that △s AFC, BFD are also similar. Find other pairs of similar △s in the same figure.

PROPOSITION 5. THEOREM.

If the sides of two triangles, taken in order, about each of their angles be proportionals, the triangles shall be equiangular to one another and shall have those angles equal which are opposite to the homologous sides.

In $\triangle$s ABC, DEF, let BA : AC :: ED : DF,
AC : CB :: DF : FE,
and ∴ BA : CB :: ED : FE, [Ex æq.
then $\angle$s A, B, C = $\angle$s D, E, F respectively.

From AB, AC, produced if necessary, cut off AH, AK equal to DE, DF, and join HK.



Then BA : AC :: DE : DF,
:: AH : AK;
∴ BA : AH :: AC : AK;
∴ BH : AH :: CK : KA;
∴ HK is ∥ to BC;
∴ $\triangle$AHK is equiangr. to $\triangle$ABC; [I. 29.
* ∴ AH : HK :: AB : BC, [VI. 4.
:: DE : EF; [Hyp.
But AH = DE; [Const.
∴ HK = EF;

* The student's attention is particularly drawn to this use of VI. 4 in proving VI. 5.

$\therefore \triangle DEF$ is equiangr. to $\triangle AHK$; [I. 8.

$\therefore \triangle DEF$ is also equiangr. to $\triangle ABC$.

From VI. 5 we see that—

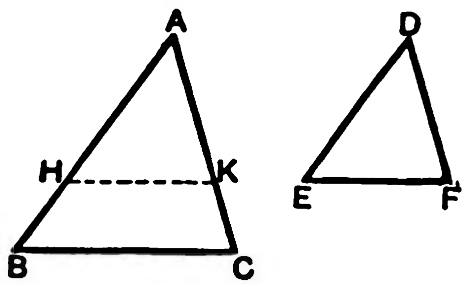
If the sides of a triangle have to one another the same ratios as the sides of a given triangle, the triangle is 'given in species'; and hence also that—

If the sides of a triangle have given ratios to one another, the triangle is 'given in species' (Euclid's *Data*, 45).

PROPOSITION 6. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular to one another and shall have those angles equal which are opposite to the homologous sides.

In $\triangle$s ABC, DEF, let $\angle$ A = $\angle$ D,
and let BA : AC :: ED : DF;
then $\angle$ B = $\angle$ E,
and $\angle$ C = $\angle$ F.



From AB, AC, produced if necessary, cut off AH, AK equal to DE, DF respectively, and join HK.

Then $\angle$s AHK, AKH = $\angle$s E, F. [I. 4.

But BA : AC :: ED : DF,
:: AH : AK;

$\therefore$ BA : AH :: AC : AK; [ALTERNANDO.

$\therefore$ BH : AH :: CK : AK; [DIVIDENDO

$\therefore$ HK is $\parallel$ to BC;

$\therefore$ $\angle$ B = $\angle$ AHK,
= $\angle$ E.

Similarly $\angle$ C = $\angle$ F.

From VI. 6 we see that—

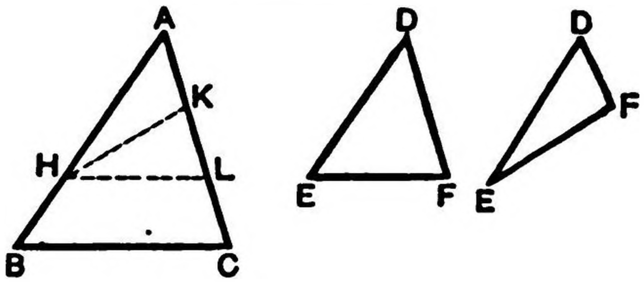
If one of the angles of a triangle be given, and if the sides about it have a given ratio to one another, the triangle is given in species (Euclid's *Data*, 44).

Ex. 694.—If, in the fig. of VI. 3, AC is a mean propl. between BC and CD, ∠ BAC is double of ∠ B.

PROPOSITION 7. Theorem.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then if each of the remaining angles be acute, or each obtuse, or if one of them be a right angle, the triangles shall be equiangular, and shall have those angles equal about which the sides are proportional.

In $\triangle$s ABC, DEF, let $\angle$ A = $\angle$ D,
and let AB : BC :: DE : EF ;
then if C and F are each acute, or each obtuse, or if either C or F is a right $\angle$,
$\triangle$ABC is equiangr. to $\triangle$DEF



From AB, AC, produced if necessary, cut off AH, AK equal to DE, DF, and join HK.

Then $\triangle$AHK = $\triangle$DEF in all respects. [I. 4.

Now if HK is ‖ to BC,
$\angle$ C = $\angle$ AKH,
= $\angle$ F.

But if HK is not ‖ to BC, draw HL ‖ to BC,
Then $\triangle$AHL is equiangr. to $\triangle$ABC ;
∴ AH : HL :: AB : BC, [VI. 4.
:: DE : EF,
but AH = DE ;
∴ HL = EF,
= HK ;
∴ $\angle$ HKL = $\angle$ HLK,
= $\angle$ C.

But $\angle AKH = \angle F$;

$\therefore \angle s\ C, F = \angle s\ HKL, AKH$

$= 2$ rt. $\angle s$.

Now if $\angle s$ C and F are both acute, or both obtuse, this is impossible; $\therefore$ HK will be $\parallel$ to BC;

and $\therefore \angle C = \angle F$.

Also if either $\angle C$ or $\angle F$ is right, the other must be, for if not

neither $\angle C = \angle F$,

nor $\angle s\ C, F = 2$ rt. $\angle s$.

Hence $\angle C = \angle F$;

and $\therefore \triangle$s ABC, DEF are equiangr.

NOTES.

We have really demonstrated the following proposition, which includes Euclid's:—

If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, *so that the sides opposite the equal angles are homologous*, **the triangles have their third angles either equal or supplementary, and in the former case the triangles are similar.**

Cor.—**Two such triangles are similar**

(1) If the two angles given equal are right angles or obtuse angles.

(2) If the angles opposite to the other two homologous sides are both acute or both obtuse, or if one of them is a right angle.

(3) If the sides opposite to the equal angles are respectively not less than the other pair of homologous sides (Syllabus).

Note that the enunciation of VI. 7 requires the insertion of the qualification in italics to render it true.

Note also that just as VI. 5 is a generalisation of I. 8, and VI. 6 is a generalisation of I. 4, so VI. 7 is a generalisation of the theorem given as Ex. 166.

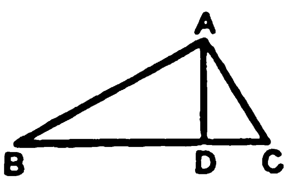
From VI. 7 we see that if the sides of a right-angled triangle about one of the acute angles have a given ratio to one another, the triangle is given in species (Euclid, *Data*, 46).

PROPOSITION 8. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to one another.

Let $\triangle$ABC have the $\angle$A a rt. $\angle$, and draw AD $\perp$r to BC; then $\triangle$s BDA, DAC shall be similar to the $\triangle$ABC and to one another.

In $\triangle$s BDA, BAC, rt. $\angle$ BDA=rt. $\angle$ BAC,
and $\angle$ B is common;
$\therefore$ third $\angle$ BAD=third $\angle$ ACB;
$\therefore$ $\triangle$BAD is equiangr. to $\triangle$BAC.



Similarly $\triangle$CAD is equiangr. to $\triangle$BAC;
$\therefore$ $\triangle$BAD is equiangr. to $\triangle$CAD.

And since they are equiangular, they have the sides about the equal angles proportionals;
$\therefore$ they are similar to one another.

COR. (1).—BD : DA :: DA : DC ($\because$ $\triangle$ABD is equiangr. to $\triangle$ADC).

(2).—CB : BA :: BA : BD ($\because$ $\triangle$ABC is equiangr. to $\triangle$ABD).

(3).—CB : CA :: CA : CD ($\because$ $\triangle$ABC is equiangr. to $\triangle$ACD).

i.e. (1) **The perpendicular DA is a mean proportional between the segments of the base.**

(2) **and** (3) **Each of the sides BA and AC is a mean**

proportional between the hypothenuse and its projection on the hypothenuse.

These corollaries are very important, and should be carefully remembered by the student.

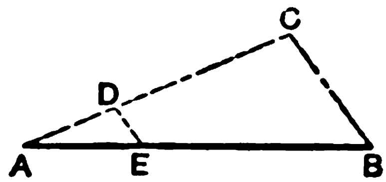
Ex. 695.—If in a $\triangle$ ABC, AD is drawn meeting BC in D, such that $\angle$ BAD = $\angle$ C, show that AB is a mean proportional between CB, BD.

Ex. 696.—BA is a tangent to a circle ADC, and BDC is a secant; show that AB is a mean proportional between CB, BD.

PROPOSITION 9. Problem.

From a given straight line to cut off any part required.

Let AB be the given st. line; it is reqd. to cut off any part from it.



Draw any st. line AD from A not in the same st. line with AB, and produce it to C, such that AC contains AD the same number of times as AB is to contain the part reqd. Join BC, and through D draw DE || to BC; then AE is the part reqd.

BE : EA :: CD : DA (∵ DE is || to BC);

∴ BA : AE :: CA : AD, [Comp.

hence BA contains AE the same number of times that CA contains AD (*i.e.*, AE is the part reqd.).

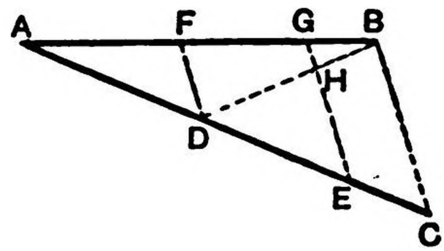
Note that the word 'part' is used in the sense of 'submultiple' or 'aliquot part,' such as one-third, one-fourth.

The *construction* could easily be extended to any fractional part whatever, such as four-sevenths or five-ninths, and the *demonstration* could be effected *by means of Book I. only*. We leave these to the student as exercises (see Ex. 251).

PROPOSITION 10. Problem.

To divide a given straight line similarly to a given divided straight line, that is, into parts which have the same ratio to one another that the parts of the given divided straight line have.

Let AC be a st. line divided in D, E, and AB a st. line which it is reqd. to divide similarly, placed with one end common, but not in the same st. line.



Join BC. Through D, E draw DF, EG ‖ to BC.
Then AB is divided in F and G as reqd.
Join DB, cutting EG in H.
Then BG : GF :: BH : HD (∵ GH is ‖ to FD),
:: CE : ED (∵ HE is ‖ to BC).
And GF : FA :: ED : DA (∵ DF is ‖ to EG);
∴ BG : FA :: CE : DA. [Ex æquali.

NOTE.

We have here demonstrated that—

If two straight lines are cut by three parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other (Syllabus).

The student should apply the method of VI. 10 to the following problems :—

(1) To divide a straight line into two parts which are to one another in the ratio of two given straight lines.

(2) To produce a straight line so that the whole line thus produced and the part produced shall be to one another in the ratio of two given straight lines.

Ex. 697.—A and B are two fixed points, and P any point such that AP : PB is a given ratio of inequality. Show that P lies on a fixed circle.

Suppose the ratio of greater inequality.

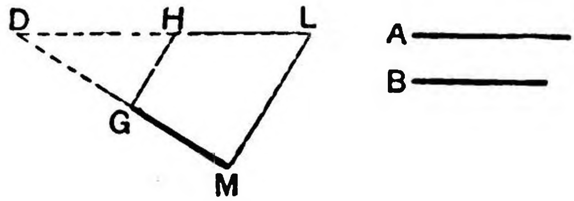
Find points C *and* D *in* AB *and* AB *produced, such that* AC : CB *and* AD : DB *in the given ratio of* AP : PB.

Then use VI. 3 *and VI. A. to show that angle* CPD *is a right angle* (*see Ex.* 16).

PROPOSITION 11. Problem.

To find a third proportional to two given straight lines.

Let A, B be the two given st. lines; it is reqd. to find a third propl. to them.



From any pt. D draw two st. lines DHL, DGM not in the same st. line, and such that DH = A, and HL = DG = B.

Join HG. Through L draw LM || to HG.

GM is the propl. reqd.

∵ HG is || to LM ;

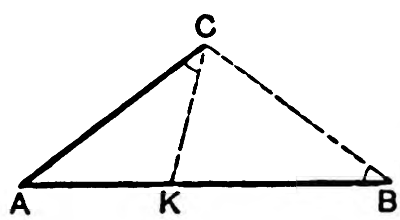
∴ DH : HL :: DG : GM ; [VI. 2.

∴ A : B :: B : GM;

i.e. GM is a third propl. to A, B.

Ex. 698.—Use VI. 8 to get another construction for a third proportional.

Ex. 699.—A circle is described to pass through B and C, and touch AC at C. If it cuts AB again at K, show that AK is a third proportional to AB, AC.

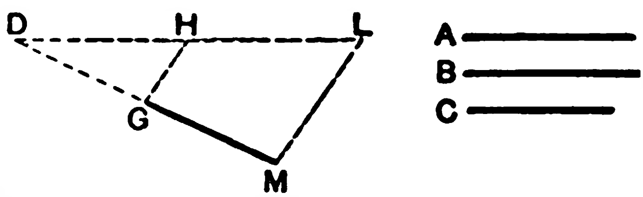


Ex. 700.—At the point C make ∠ACK = ∠ABC, and let CK meet AB, produced if necessary, at K; then AK is a mean proportional between AB, AC.

PROPOSITION 12. PROBLEM.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is reqd. to find a fourth proportional to them.



From any pt. D draw two st. lines DHL, DGM not in the same st. line, so that DH = A, HL = B, DG = C.

Join HG. Through L draw LM ∥ to HG.

GM is the propl. reqd.

∵ HG is ∥ to LM,

∴ DH : HL :: DG : GM ; [VI. 2

∴ A : B :: C : GM,

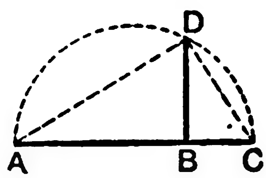
i.e. GM is the fourth propl. to A, B, C.

PROPOSITION 13. PROBLEM.

To find a mean proportional between two given straight lines.

Let AB, BC be two given st. lines; it is reqd. to find a mean propl. between them.

Place them so that ABC is a st. line.



On AC describe a semi-⊙ ADC.

Draw BD ⊥r to AC.

BD is the propl. reqd.

Join AD, DC.

Then ∠ ADC is a rt. ∠, and BD is ⊥ to AC;

∴ AB : BD :: BD : BC, [VI. 8. COR.

i.e. BD is a mean propl. between AB, BC

EX. 701.—ABC is a straight line. A tangent AP is drawn to *any* circle through B and C. Show AP is a mean propl. between AB and AC.

Join BP, CP, *and use III.* 32 *and VI.* 4.

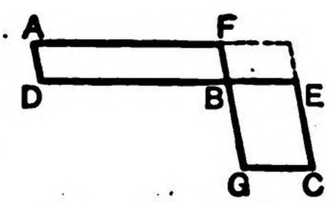
Def.—**Reciprocal figures, viz. triangles and parallelograms, are such as have their sides about two of their angles proportionals in such a manner that a side of the first figure is to a side of the other as the remaining side of this other is to the remaining side of the first.**

PROPOSITION 14. Theorem.

(1) **Equal parallelograms which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional.**

(2) Conversely:—**Parallelograms which have one angle of the one equal to one angle of the other and the sides about the equal angles reciprocally proportional are equal.**

(1) Let AB, BC be equal ||gms, having their ∠s at B equal; then DB : BE :: GB : BF.



Let DB, BE be placed in the same st. line; then GB, BF are in the same st. line. [I. 14.

Complete the ||gm EF.

∵ ||gm AB = ||gm BC;

∴ ||gm AB : ||gm EF :: ||gm BC : ||gm EF. [V. 7.

But ||gm AB : ||gm EF :: DB : BE,
and ||gm BC : ||gm EF :: GB : BF; } [VI. 1.

∴ DB : BE :: GB : BF. [V. 11.

(2) Next let ||gms AB, BC have their ∠s at B equal, and let DB : BE :: GB : BF;

then ||gm AB = ||gm BC.

With the same construction

$$\left.\begin{array}{r}\|\text{gm } AB : \|\text{gm } EF :: DB : BE, \\ \text{and } \|\text{gm } BC : \|\text{gm } EF :: GB : BF.\end{array}\right\}$$ [VI. 1.

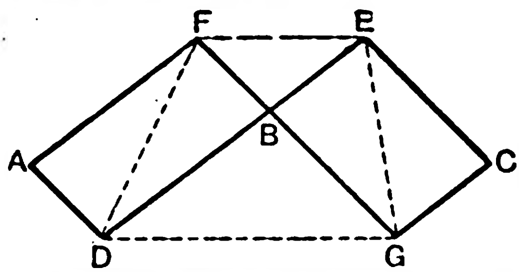
But $DB : BE :: GB : BF$;

$\therefore \|\text{gm } AB : \|\text{gm } EF :: \|\text{gm } BC : \|\text{gm } EF$;

$\therefore \|\text{gm } AB = \|\text{gm } BC$. [V. 9.

Alternative Proof—

(1) Let AB, BC be equal $\|$gms having their $\angle$s at B equal; then $DB : BE :: GB : BF$.



Let DB, BE be placed in the same st. line, then GB, BF are in the same st. line. [I. 14.

Join EF, FD, DG, GE.

$\because \|\text{gm } AB = \|\text{gm } BC$;

$\therefore \triangle DBF = \triangle GBE$;

$\therefore \triangle DEF = \triangle GFE$;

$\therefore EF$ is $\|$ to DG;

$\therefore DE : EB :: GF : FB$; [VI. 2.

$\therefore DB : BE :: GB : BF$. [Dividendo.

(2) Next let $\|$gms AB, BC have their $\angle$s at B equal, and let $DB : BE :: GB : BF$,

then $\|\text{gm } AB = \|\text{gm } BC$.

With the same construction

$\because DB : BE :: GB : BF$;

$\therefore DE : EB :: GF : FB$; [Componendo.

$\therefore EF$ is $\|$ to DG;

$\therefore \triangle DEF = \triangle GFE$;

$\therefore \triangle DBF = \triangle GBE$;

$\therefore \|\text{gm } AB = \|\text{gm } BC$.

NOTE.

With reference to the definition of *Reciprocal Figures*, Simson remarks that it seems to be due to some unskilful editor.—'For there is no mention made by Euclid, nor, as far as I know, by any other geometer, of reciprocal figures.'

Ex. 702.—Show that B lies on the diagonal of the parallelogram formed by producing AF, CE, CG, AD to meet, and hence demonstrate each part of VI. 14.

Ex. 703.—(i.) If the rectangle contained by two adjacent sides of one parallelogram be equal to the rectangle contained by two adjacent sides of another which is equiangular to it, the parallelograms shall be equal to one another.

(ii.) Conversely :—If two equiangular parallelograms be equal to one another, the rectangle contained by two adjacent sides of one of them will be equal to that contained by two adjacent sides of the other.

Ex. 704.—Equal parallelograms which have the sides about an angle in each reciprocally proportional are equiangular to one another.

Ex. 705.—In the fig. of VI. 14, B and the mid points of EF and DG are collinear.

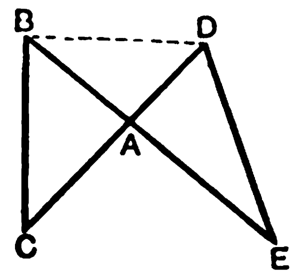
Ex. 706.—In the fig. of VI. 14, if AC cuts DF and GE in H and K, AH = CK.

PROPOSITION 15. THEOREM.

(1) **Equal triangles which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional.**

(2) Conversely:—**Triangles which have one angle of the one equal to one angle of the other and their sides about the equal angles reciprocally proportional are equal to one another.**

(1) Let ABC, ADE be equal $\triangle$s, having their $\angle$s at A equal; then CA : AD :: EA : AB.



Let CA, AD be placed in the same st. line; then EA, AB are also in a st. line. [I. 14.

Join BD.

$\because \triangle ABC = \triangle ADE$;

$\therefore \triangle ABC : \triangle ABD :: \triangle ADE : \triangle ABD$. [V. 7.

$\left.\begin{array}{l}\text{But } \triangle ABC : \triangle ABD :: CA : AD, \\ \text{and } \triangle ADE : \triangle ABD :: EA : AB;\end{array}\right\}$ [VI. 1.

$\therefore CA : AD :: EA : AB$. [V. 11.

(2) Next let $\triangle$s ABC, ADE have their $\angle$s at A equal, and let CA : AD :: EA : AB;

then $\triangle ABC = \triangle ADE$.

With the same construction,

$\left.\begin{array}{l}\triangle ABC : \triangle ABD :: CA : AD, \\ \text{and } \triangle ADE : \triangle ABD :: EA : AB.\end{array}\right\}$ [VI. 1.

But CA : AD :: EA : AB; [Hyp.

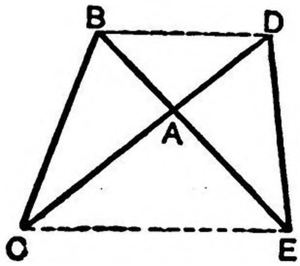
$\therefore \triangle ABC : \triangle ABD :: \triangle ADE : \triangle ABD$;

$\therefore \triangle ABC = \triangle ADE$. [V. 9.

Alternative Proof—

(1) Let ABC, ADE be equal $\triangle$s, having the $\angle$s at A equal; then CA : AD :: EA : AB.

Let the $\triangle$s be placed so that CA, AD are in a st. line; then EA, AB are also in a st. line. [I. 14.



Join BD, CE.

$\because$ $\triangle$BAC=$\triangle$DAE;

$\therefore$ $\triangle$BDC=$\triangle$DBE;

$\therefore$ CE is $\parallel$ to BD;

$\therefore$ CD : DA :: EB : BA; [VI. 2.

$\therefore$ CA : AD :: EA : AB. [Dividendo.

(2) Next let $\triangle$s ABC, ADE have the angles at A equal, and let CA : AD :: EA : AB;

then $\triangle$ABC=$\triangle$ADE.

With the same construction

$\because$ CA : AD :: EA : AB;

$\therefore$ CD : DA :: EB : BA; [Componendo.

$\therefore$ CE is $\parallel$ to BD;

$\therefore$ $\triangle$BDC=$\triangle$DBE;

$\therefore$ $\triangle$BAC=$\triangle$DAE.

Ex. 707.—Demonstrate VI. 15 by means of VI. 14.

Ex. 708.—(i.) If two triangles have one angle of the one equal to one angle of the other, and the rectangle contained by the sides about one of these angles equal to the rectangle contained by the sides about the other, the triangles are equal.

(ii.) Conversely :—If two triangles have one angle of the one equal to one angle of the other, the rectangle contained by the sides about one

of those angles is equal to the rectangle contained by the sides about the other.

Ex. 709.—Equal triangles which have the sides about one angle in each reciprocally proportional have those angles either equal or supplementary.

Ex. 710.—In the figure of VI. 15, draw through A a parallel to BD, meeting BC, DE in H, K; and show that HA=AK, both with and without the use of proportion.

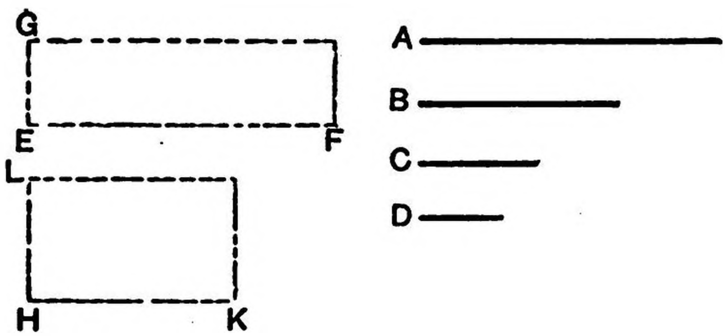
Ex. 711.—In the figure of VI. 15, A and the mid points of BD and CE are collinear.

PROPOSITION 16. THEOREM.

(1) If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

(2) Conversely :—**If the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.**

(1) Let A, B, C, D be four st. lines, such that A : B :: C : D, then rect. A, D = rect. B, C.



Draw two st. lines EF, EG ⊥r to one another and equal to A, D respectively, and two st. lines HK, HL ⊥ to one another and equal to B, C respectively, and complete the rects. FG, KL.

∵ A : B :: C : D;
∴ EF : HK :: HL : EG,
and rt. ∠ E = rt. ∠ H;
∴ FG = KL,
i.e. rect. EF, EG = rect. HK, HL;
∴ rect. A, D = rect. B, C.

(2) Let rect. A, D = rect. B, C,
then A : B :: C : D,

With the same construction
∵ rect. A, D = rect. B. C;

$\therefore$ rect. EF, EG = rect. HK, HL,
i.e. FG = KL,
and rt. $\angle$ E = rt. $\angle$ H;
$\therefore$ EF : HK :: HL : EG;
$\therefore$ A : B :: C : D.

Ex. 712.—Through any point O draw straight lines POS, QOR such that

OP, OQ, OR, OS = A, B, C, D,
and deduce VI. 16 by showing (1) that
if A : B :: C : D,
$\triangle$s OPQ, OSR are equiangr. [VI. 6.
Hence P, Q, R, S are concyclic,
and rect. OP, OS = rect. OQ, OR;
(2) if A, D = B, C,
P, Q, R, S are concyclic.
Hence $\triangle$s OPQ, OSR are equiangr.
and OP : OQ :: OR : OS.

Also prove when OS, OR are taken along OP, OQ.

PROPOSITION 17. Theorem.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean.

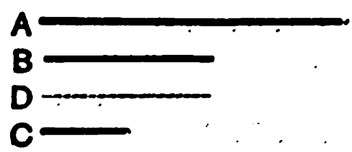
Conversely :—**If the rectangle contained by the extremes is equal to the square on the mean, the three straight lines are proportionals.**

(i.) Let A, B, C be three st. lines, such that

A : B :: B : C,

then rect. A, C = sq. on B.

Draw a st. line D equal to B.



∵ A : B :: B : C;

∴ A : B :: D : C;

∴ rect. A, C = rect. B, D,

= sq. on B.

(ii.) Let rect. A, C = sq. on B,

then A : B :: B : C.

With the same construction

rect. A, C = sq. on B,

= rect. B, D;

∴ A : B :: D : C;

∴ A : B :: B : C.

Ex. 713.—From any point O draw two lines OPS, OQ, not in the same st. line, such that OP, OQ, OS = A, B, C,

and demonstrate VI. 17 by showing

(1) that if A : B :: B : C,

△s OPQ, OQS are equiangr.

Hence OQ touches circum-⊙ of △ PQS,

and rect. OP, OS = sq. on OQ;

(2) that if rect. A, C = sq. on B,

OQ touches circum-⊙ of △ PQS.

Hence △s OPQ, OQS are equiangr.

and OP : OQ :: OQ : OS.

Def.—If two straight lines are homologous sides of two similar rectilineal figures, the figures are said to be 'similarly situated' or 'similarly described' on those straight lines.

PROPOSITION 18. Problem.

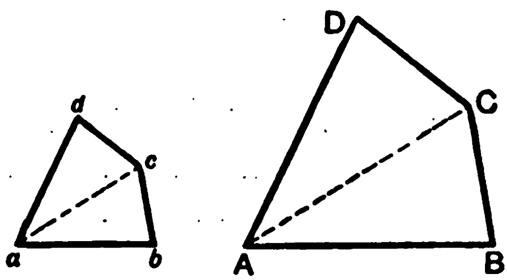
Upon a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let ab be the given st. line, and ABCD the given rectilineal figure; it is reqd. to describe upon ab a fig. similar and similarly situated to ABCD.

Join AC, and at a, b make the $\angle$s bac, abc equal to $\angle$s BAC, ABC resp., then $\angle$ bca $=$ $\angle$ BCA. [I. 32.

At a, c make $\angle$s cad, acd equal to $\angle$s CAD, ACD resp., then $\angle$ d $=$ $\angle$ D; [I. 32.

then fig. abcd is similar and similarly situated to fig. ABCD.



$\because$ $\triangle$s abc, ABC are equiangr.;

$\therefore$ ab : ac :: AB : AC.
Similarly ac : ad :: AC : AD, } [VI. 4.

$\therefore$ ab : ad :: AB : AD. [Ex æquali.

Similarly bc : cd :: BC : CD.

Also ab : bc :: AB : BC,
and cd : da :: CD : DA. } [VI. 4.

Again $\because$ $\angle$s bac, cad $=$ $\angle$s BAC, CAD;

$\therefore$ $\angle$ bad $=$ $\angle$ BAD.

Similarly $\angle$ bcd $=$ $\angle$ BCD;

$\therefore$ $\angle$s at a, b, c, d $=$ corresp. $\angle$s at A, B, C, D,

and sides about a, b, c, d are propl. to sides about A, B, C, D.

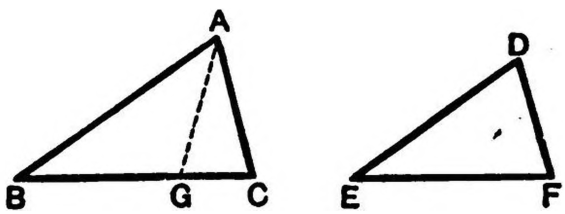
$\therefore$ abcd is similar to ABCD, and is situated on ab similarly to ABCD on AB.

The method can easily be extended to figures of five or more sides. Simson added an extension to pentagons. We follow Euclid in leaving this to the student.

PROPOSITION 19. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let △s ABC, DEF have ∠s A, B, C equal to ∠s D, E, F respectively, so that BC is homologous to EF;
then △ABC : △DEF in the duplicate ratio of BC : EF.



On BC take BG, such that BC : EF :: EF : BG, [VI. 11.
and ∴ BC : BG in duplicate ratio of BC : EF. [V. DEF. 10.
Now AB : BC :: DE : EF; [VI. 4.
∴ AB : DE :: BC : EF, [ALT.
:: EF : BG.
But ∠B = ∠E;
∴ △ABG = △DEF; [VI. 15.
∴ △ABC : △DEF :: △ABC : △ABG,
:: BC : BG, [VI. 1.
i.e. △ABC : △DEF in duplicate ratio of BC : EF.

COR.—From this it is plain that

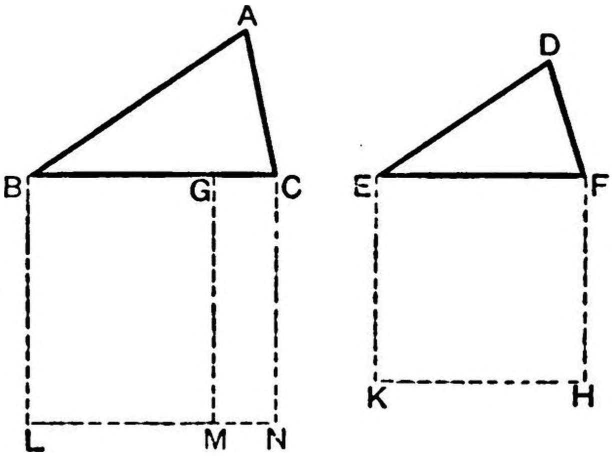
If three straight lines be proportionals, as the *first* is to the *third*, so is any triangle upon the *first* to the similar and similarly described triangle upon the *second*.

NOTE.

It may be easily deduced that :—

Similar triangles ABC, DEF are as the squares of their homologous sides BC, EF.

Let BG be a third propl. to BC, EF as in VI. 19.
On BC describe a sq. BCNL, and draw GM || to BL or CN.
Then BM = rect. BL, BG,
= rect. BC, BG,
= sq. on EF.



Now $\triangle$ ABC : $\triangle$ DEF :: BC : BG,
:: BN : BM,
:: sq. on BC : sq. on EF.

Hence *if* X *and* Y *be two straight lines*, the '**duplicate ratio of X to Y**' may always be taken as that of '**the square on X to the square on Y.**'

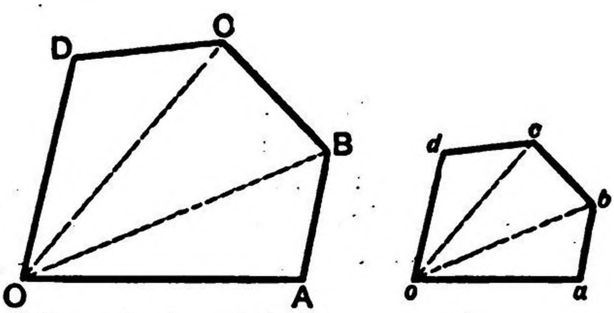
N.B.—*The student must be careful not to confuse the term 'similarly situated' or 'similarly described' with the term 'similarly placed' which has a much more restricted meaning* (see p. 450).

Ex. 714.—In the fig. of VI. 8, BD : DC :: sq. on BA : sq. on AC.

PROPOSITION 20. THEOREM.

Similar polygons may be divided into the same number of similar triangles which are to one another in the same ratio as the polygons are; and the polygons are to one another in the duplicate ratio of their homologous sides.

Let OABCD, oabcd be similar polygons, and let OA be homologous to oa; then OABCD, oabcd may be divided into the same number of similar △s which are to one another in the same ratio as OABCD to oabcd; and OABCD : oabcd in the dupl. ratio of OA to oa.



Join OB, OC, ob, oc.
∵ OABCD, oabcd are similar;
∴ ∠OAB = ∠oab,
and OA : AB :: oa : ab;
∴ OAB, oab are similar triangles such that
∠OBA = ∠oba,
and OB : BA :: ob : ba. [VI. 6.
But ∠ABC = ∠abc,
and AB : BC :: ab : bc; ∵ OABCD, oabcd are similar;
∴ remg. ∠OBC = remg. ∠obc,
and OB : BC :: ob : bc;
∴ OBC, obc are similar △s.
Similarly OCD, ocd are similar △s.

Again $\triangle OAB : \triangle oab$ in dupl. ratio of $OB : ob$,
and $\triangle OBC : \triangle obc$ in dupl. ratio of $OB : ob$;
$\therefore \triangle OAB : \triangle oab :: \triangle OBC : \triangle obc$.
Similarly $\triangle OBC : \triangle obc :: OCD : ocd$;
$\therefore$ sum of $\triangle$s OAB, OBC, OCD : sum of $\triangle$s oab, obc, ocd :: $\triangle OAB$: oab, [V. 12.
i.e. OABCD : oabcd :: $\triangle OAB : \triangle oab$,
i.e. in dupl. ratio of $OA : oa$.

COR. i.—**Similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.**

COR. ii.—If to the homologous sides BC, EF of two similar rectilineal figures P and Q a third proportional BG be taken, as in VI. 19,

BC : BG in the dupl. ratio of BC : EF;
$\therefore$ BC : BG :: P : Q.

Hence generally :—

If three straight lines be proportionals, as the first is to the third, so is any rectilineal figure on the first to the similar and similarly described rectilineal figure on the second.

COR. iii.[1]—**Similar rectilineal figures are to one another as the squares on their corresponding sides.**

Ex. 715.—**oabcd can be applied to OABCD, so that oa, ob, oc, od fall along OA, OB, OC, OD, while ab, bc, cd are parallel to AB, BC, CD.**

This exercise is important as the forerunner of a more general theorem, viz.—

If two similar rectilineal figures be placed with their homologous

[1] This has important applications in Practical Geometry.

sides parallel, the lines joining their corresponding corners are either parallel or concurrent.

Some attention will be given to this in an appendix on **Similarity**, p. 448.

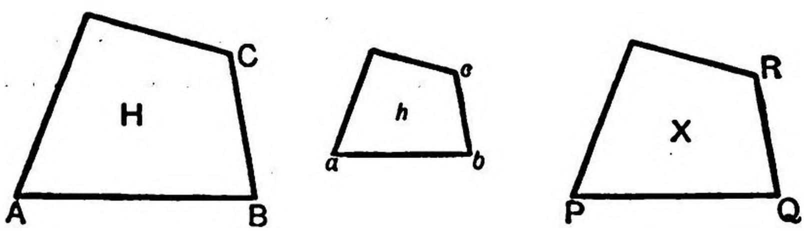
Its application is of great use in Practical Geometry.

Ex. 716.—A trapezoid is divided into two similar trapezoids by a ∥ to its ∥ sides. Show that they are equivalent respectively to the two △s into which the whole trapezoid is divided by either diagl.

PROPOSITION 21. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure are also similar to one another.

Let each of the rectl. figs. H, h be similar to the rectl. fig. X; they shall be similar to one another.



Let the sides AB, BC of H, and the sides ab, bc of h be homologous to the sides PQ, QR of X.

Then $\angle$ B = $\angle$ Q
= $\angle$ b,
and AB : BC :: PQ : QR,
:: ab : bc.

Similarly the other $\angle$s of H and h are equal, and the sides about them are propl.;

$\therefore$ H and h are similar to one another.

PROPOSITION 22. THEOREM.

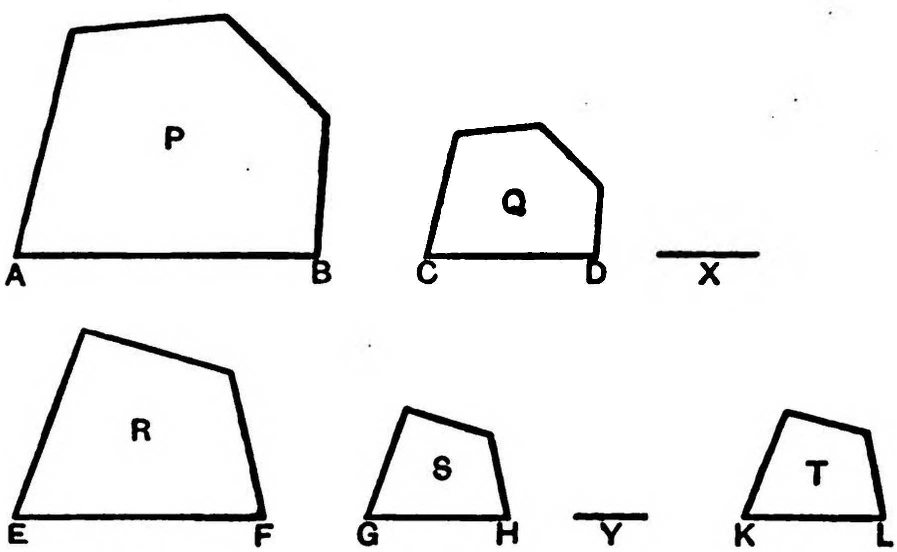
(1) If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals.

(2) Conversely :—If the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

(1) Let the four straight lines AB, CD, EF, GH be such that AB : CD :: EF : GH ;

Let P and Q be similar rectl. figs. similarly described on AB, CD, and let R and S be similar rectl. figs. similarly described on EF, GH ;

then P : Q :: R : S.



To AB, CD take a third propl. X,
and to EF, GH take a third propl. Y.
Then CD : X :: AB : CD,
:: EF : GH,
:: GH : Y;
but AB : CD :: EF : GH ;

$\therefore AB : X :: EF : Y.$ [Ex æquali.

Again $AB : X :: P : Q,$ and $EF : Y :: R : S;$ [VI. 20, Cor. ii.

$\therefore P : Q :: R : S.$

(2) Next, the figs. P, Q, R, S being described on AB, CD, EF, GH as in (1), let $P : Q :: R : S$, then $AB : CD :: EF : GH$.

Take a fourth propl. KL to AB, CD, EF, and on KL describe a rectl. fig. T similar and similarly situated to R or S.

$$\because AB : CD :: EF : KL;$$
$$\therefore P : Q :: R : T;$$
$$\text{but } P : Q :: R : S;$$
$$\therefore R : S :: R : T;$$
$$\therefore S = T;$$
$$\therefore GH = KL;$$
$$\therefore AB : CD :: EF : GH.$$

NOTES.

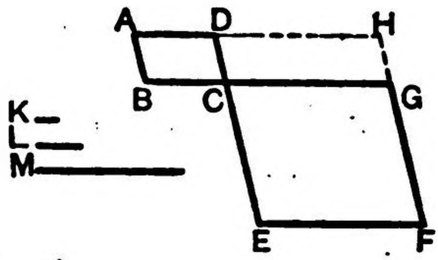
(1) We have demonstrated a more general proposition than VI. 22.

(2) An assumption has been made in the demonstration of (ii). We leave it to the student to enunciate generally the proposition assumed, and to demonstrate it.

PROPOSITION 23. THEOREM.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangr. ||gms such that ∠ BCD = ∠ ECG; then ||gm AC : ||gm CF in the ratio compounded of the ratios BC : CG and DC : CE.



Let BC, CG be placed in a st. line; then DC and CE are also in a st. line. [I. 14.

Complete the ||gm DG.

Take any st. line K and two others L, M,

such that BC : CG :: K : L,

and DC : CE :: L : M;

then the ratio K : M is compounded of the ratios BC : CG, and DC : CE (Def. of Comp. Ratio).

Now ||gm AC : ||gm DG :: BC : CG, [VI. 1.

:: K : L.

Similarly ||gm DG : ||gm CF :: L : M;

∴ ||gm AC : ||gm CF :: K : M, [Ex æquali.

i.e. in the ratio compounded of the ratios

BC : CG and DC : CE.

NOTE.

Since all rectangles are 'equiangular parallelograms,' it follows that **'Rectangles are to one another in the ratio compounded of the ratios of their sides.'**

Hence :—*If* A, B, X, Y *be four straight lines*, **the ratio compounded of the ratios A : X, B : Y is that of rect. A, B : rect. X, Y.**

Hence also if ABCD, ECGF be equiangular parallelograms as in VI. 23, ||gm AC : ||gm CF :: rect. BC, CD : rect. EC, CG.

Ex. 717.—**Triangles which have one angle of the one equal to one angle of the other are to one another in the ratio compounded of the ratios of the sides containing the equal angles.**

Ex. 718.—**Triangles which have one angle of the one supplementary to one angle of the other are to one another in the ratio compounded of the ratios of the sides containing the supplementary angles.**

Ex. 719.—If ∠ABC + ∠DEF = 2 rt. ∠s, show that

△ABC : △DEF :: rect. AB, BC : rect. DE, EF.

Ex. 720.—O is the centre of the in-⊙ of △ABC; L, M, N its pts. of contact with BC, CA, AB; show that

△MON : △NOL : △LOM :: BC : CA : AB.

Compare each with the whole △ABC.

Ex. 721.—If ∠ABC = ∠DEF, show that

△ABC : △DEF :: rect. AB, BC : rect. DE, EF.

Hence deduce VI. 19.

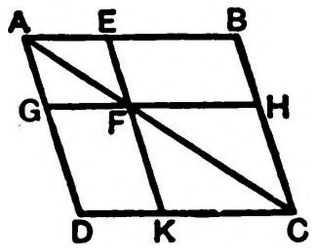
Ex. 722.—Deduce VI. 14 from VI. 23.

Ex. 723.—If in the figure of VI. 23 AB, HC are produced to meet in P, show that ||gm AC : ||gm CF :: BP : CE.

PROPOSITION 24. Theorem.

The parallelograms about the diameter of a parallelogram are similar to the whole and to one another.

Let EG, HK be ||gms about the diagl. AC of the ||gm ABCD; they shall be similar to ||gm ABCD and to one another.



∵ GH is || to DC;
∴ ext. ∠ AGF = int. ∠ D.
Similarly ext. ∠ AEF = int. ∠ B.
Also ∠ EFG = opp. ∠ EAG,
= opp. ∠ BCD;
∴ ||gms. EG, BD are equiangr.
Again ∵ ∠ AGF = ∠ D,
and ∠ GAF is common to △s AGF, ADC;
∴ they are equiangr.;
∴ AG : GF :: AD : DC;

$$\left.\begin{array}{r} \therefore\ \text{FE : EA :: CB : BA,} \\ \text{EF : FG :: BC : CD,} \\ \text{and AG : AE :: AD : AB;} \end{array}\right\} \because \text{ opp. sides of ||gms are equal;}$$

∴ ||gm EG is similar to ||gm BD.
Similarly ||gm HK is similar to ||gm BD;
∴ also ||gms EG, HK are similar to one another.

NOTE.

A particular case of this proposition has been demonstrated in II. 4, and is stated as a corollary. See p. 130.

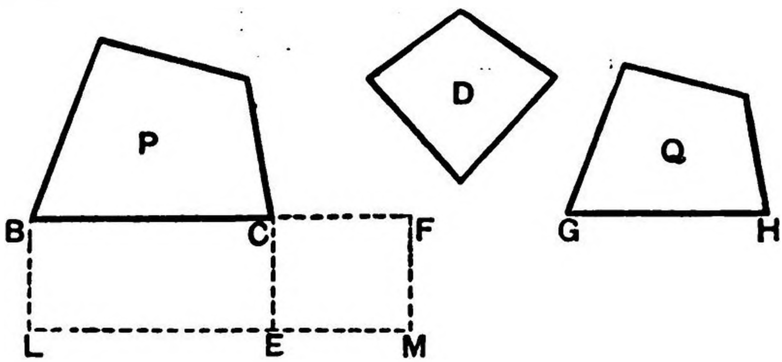
Ex. 724.—Prove that parallelogram HK is similar to parallelogram EG by 'dividendo.'

Ex. 725.—Prove first that parallelogram EG is similar to parallelogram HK, and hence by 'componendo' to parallelogram BD.

PROPOSITION 25. PROBLEM.

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.

Let it be reqd. to describe a rectl. fig. similar to P and equal to D.



To the side BC of P apply the rectangle BCEL equal to P; and to CE apply the rectangle CEMF equal to D. [I. 45, COR.

Then BC, CF are in a st. line, and also LE, EM.

Find a mean propl. GH between BC, CF, and on GH describe a rect. fig. Q similar and similarly situated to P. Q is the rectl. fig. reqd.

Now BE : CM :: BC : CF,
:: P : Q. [VI. 20, COR. ii.
But P=BE;
∴ Q=CM,
=D,
and Q is similar to P;
∴ Q is the rectl. fig. reqd.

NOTE.

VI. 23 might be enunciated in more familiar language, thus :—

To describe a rectilineal figure **of the same shape** as one given rectilineal figure and **of the same size** as another.

Or more briefly :—

To describe a rectilineal figure **of given shape and size.**

Such problems as 'To describe an equilateral triangle equal to a given square' are merely special cases of the general proposition, and may be solved by Euclid's method. It sometimes happens that a special solution may be obtained shorter than this. For example :—

Ex. 726.—Two triangles ABC, ABG have a common angle B and a common side AB ; required to draw a triangle similar to ABC and equal to ABG.

Find a mean propl. EF *between* BC *and* BG, *and describe a* △ DEF *on it similar to* ABC.

Ex. 727.—Construct an isosceles triangle equal in area to a given scalene triangle, and having a common vertical angle.

Ex. 728.—Bisect a triangle by a straight line drawn parallel to its base.

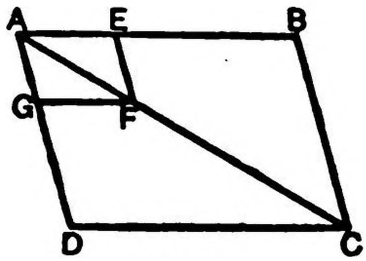
It is worthy of notice that VI. 25 contains the solution of the following :—

Ex. 729.—**To find two straight lines which have the same ratio to one another as two given rectilineal figures.**

PROPOSITION 26. THEOREM.

If two similar parallelograms have a common angle and be similarly situated, they are about the same diagonal.

Let the ||gms ABCD, AEFG be similar and similarly situated, and have $\angle$ DAB common ; then they are about the same diamr.



Join AC, AF.

In $\triangle$s AGF, ADC,

$\angle$ AGF = $\angle$ ADC, } [Hyp.

and AG : GF :: AD : DC ; }

$\therefore$ $\angle$ GAF = $\angle$ DAC ; [VI. 6.

$\therefore$ AF falls along AC,

i.e. ||gms AEFG, ABCD are about the same diamr.

NOTE.

This is a particular case of the more general proposition given on p. 392.

The straight lines drawn through the pairs (B, E), (D, G) of 'corresponding points' of the similar and similarly situated figures ABCD, AEFG meet in a point A ; hence the straight line drawn through the other pair (C, F) of corresponding points also passes through A.

In such a case the point of concurrence, A, of the straight lines drawn through corresponding points is called **a centre of similitude** of the two similar figures.

Ex. 730.—Show that in the figure of VI. 26 the other diagonals EG, BD are parallel.

Ex. 731.—Two similar triangles EFG, BCD are placed with the sides EF, FG, GE of the one parallel to the corresponding sides BC, CD, DB

of the other. If BE, DG, when produced, meet in A, show that the straight line through C, F also passes through A (*i.e.* that A is a *centre of similitude* of the two similar triangles).

Prove by ex æquali that AG : GF :: AD : CD, *and then proceed as in VI.* 26.

Ex. 732.—What other converse has VI. 24 besides VI. 26. Adapt the demonstration of VI. 26 to it.

Ex. 733.—If in the fig. of VI. 26 a straight line APQ be drawn, cutting EF, BC, or FG, CD in P, Q; show that the ratio AP : AQ is the same for all directions of the line APQ. How could you extend this to the figure of Ex. 731.

Ex. 734.—Prove VI. 26 *indirectly* (*Euclid's own method*).

It is usual to omit VI. 27, 28, 29 from the student's course of reading. The subjoined enunciations as given by Simson may perhaps suggest a reason for the omission.

27. *Of all parallelograms applied to the same straight line and deficient by parallelograms similar and similarly situated to that which is described upon the half of the line, that which is applied to the half and is similar to its defect is the greatest.*

28. *To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogram similar to a given parallelogram: but the given parallelogram to which the parallelogram to be applied is to be equal must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.*

29. *To a given straight line to apply a parallelogram equal to a given rectilineal figure exceeding by a parallelogram similar to another given.*

The following exercises, which are special cases given by Simson, may help the student to understand these somewhat obscure enunciations.

Ex. 735.—To find a point D in a given straight line AB such that rect. AD, DB = a given square which is not greater than that on half AB.

If, using the fig. of II. 5, D were the reqd. point, AH would be equal to the given square, and the problem stated in Euclid's way would be :—

To a given straight line (AB) to apply a rectangle (AH) equal to a given square not greater than that on half the line (CF) deficient by a square (DM).

For the solution use II. 5 *and I.* 4.

Ex. 736.—To find a point in a given straight line such that the rect-

angle contained by its segments shall be equal to a given rectangle not greater than the square on half the line.

Enunciate this in Euclid's manner.

For the solution use II. 14, *II.* 5, *and I.* 47.

Ex. 737.—To produce a given straight line AB to a point D, such that rect. AD, DB = a given rectangle.

With the fig. of II. 6 the problem could be stated in Euclid's way, thus:—

To a given straight line (AB) to apply a rectangle (AM) equal to a given rectangle exceeding by a square (BM).

For the solution use II. 14, *II.* 6, *and I.* 47. *See Ex.* 504.

Def.—**A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment as the greater segment is to the less.**

PROPOSITION 30. Problem.

To cut a given straight line in extreme and mean ratio.

Let AB be the given st. line; it is reqd. to cut it in extreme and mean ratio.

Divide AB at C so that rect. AB, BC=sq. on AC. [II. 11.

Then AB is cut at C in extreme and mean ratio.

∵ rect. AB, BC=sq. on AC;

∴ AB : AC :: AC : BC, [VI. 17.

i.e. AB is cut in extreme and mean ratio.

Ex. 738.—If, in the figure of VI. 13, AC is divided in extreme and mean ratio at B, and AD, produced to meet the tangent at C in F, show that CF is a mean proportional between AC, CD.

Ex. 739.—Construct a right-angled △ BAC (fig. of VI. 8), so that AC, one of the sides containing the rt. ∠, may be equal to the segt. BD of the hypotenuse BC cut off by the perpr. AD.

Hence describe a right-angled △ whose sides shall be in continued proportion.

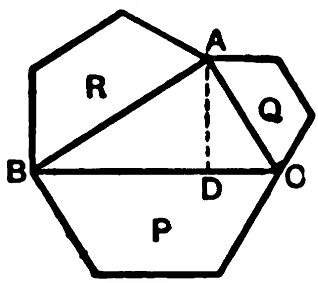
PROPOSITION 31. THEOREM.

In a right-angled triangle any rectilineal figure described upon the side opposite the right angle is equal to the similar and similarly described figures upon the sides containing the right angle.

Let P, Q, R be similar rectl. figs. similarly described on the sides BC, CA, AB of a △ABC, having ∠ BAC a rt. ∠ then Q, R together = P.

Draw AD ⊥r to BC;

then BC : BA :: BA : BD; [VI. 8, Cor.



∴ P : R :: BC : BD;

∴ R : P :: BD : BC.

Similarly Q : P :: DC : BC;

∴ R, Q together : P :: BD, CD together : BC.

But BD, CD together = BC.

∴ Q, R together = P.

Alternative Proof—

R : P in dupl. ratio of BA : BC,

and sq. on BA : sq. on BC in dupl. ratio of BA : BC;

∴ R : P :: sq. on BA : sq. on BC.

Similarly Q : P :: sq. on AC : sq. on BC;

∴ R, Q together : P :: sqs. on BA, AC : sq. on BC.

But sqs. on BA, AC = sq. on BC;

∴ Q, R together = P.

NOTE.

This theorem is a generalisation of I. 47. Special cases, capable of special demonstrations, will perhaps suggest themselves to the student as exercises.

As an example, we give the following theorem discovered by Torricelli: 'Ignorans adhuc universalem propositionem trigesimam primam de similibus figuris ab Euclide in Elementorum libro VI. allatam' (1668):—

Ex. 740.—The equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described upon the other two sides.

Let BLC, CMA, ANB *be the equilateral* △*s described externally;* O *the mid pt. of the hypotenuse* BC. *Then it is easy to show that* ON *is* ‖ *to* AC, *and hence that quadl.* ANBO = △ BNC. [Comp. Ex. 78.
= △ ABL. [Comp. I. 47.
Similarly quadl. CMAO = △ ACL.
Add equals and take away △ BAC.

The same demonstration applies to isosceles triangles, with equal vertica. angles, described on BC, CA, AB as bases.

Ex. 741.—If similar triangles BLC, CMA, ANB be similarly described on the sides of the right-angled triangle BAC (fig. VI. 31), and L be joined to the foot D of the perpendicular AD; show that triangle ANB = triangle BLD.

When the triangles BLC, CMA, ANB are equilateral, it will be an interesting exercise for the student to try to demonstrate the above property as Torricelli did, *without the use of proportion.*

As in Ex. 740 *quadl.* ANBO = △ ABL;
∴ △ ANB = *quadl.* AOBL.
= △ BDL.

PROPOSITION 32. THEOREM.

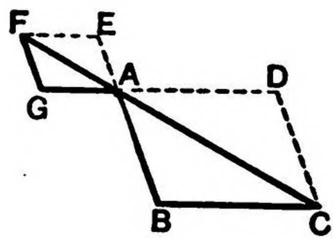
If two triangles which have two sides of the one proportional to two sides of the other be joined at one angle, so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

Let FGA, ABC be two $\triangle$s such that

FG : GA :: AB : BC,

and let FG, GA be || to AB, BC respectively;

then FA, AC are in the same st. line.



Produce BA to meet the || through F to GA or BC in E;

then $\angle$ AEF=alt. $\angle$ ABC.

Also AE : EF :: FG : GA,

:: AB : BC; [Hyp.

$\therefore$ $\angle$ EAF= $\angle$ BAC; [VI. 6.

$\therefore$ FA is in a st. line with AC. [I. 14.

NOTES.

This proposition is condemned by some writers as useless.

Mr. Todhunter points out that it is only true when the bases FA, AC lie on *corresponding sides* of the parallels.

Possibly Simson is right in his conjecture that VI. 32 was used to demonstrate the following proposition, which includes VI. 26 :—

Ex. 742.—If two similar and similarly placed parallelograms have an angle common to both, or vertically opposite angles, their diameters are in the same straight line.

Demonstrate by means of VI. 32.

It may be noticed that we have here another special case of the general theorem given on p. 392.

For if we complete the parallelogram ABCD, it can be shown that it is similar to AEFG, and A is a *centre of similitude* of the two similar rectilineal figures ABCD, AEFG.

Here any two corresponding points lie on opposite sides of A, and A is called an *internal centre of similitude.*

In VI. 26 any two corresponding points lie on the same side of A, and A is called an 'external centre of similitude' of the similar parallelograms ABCD, AEFG.

Ex. 743.—Two similar triangles EFG, BCD are placed with the sides EF, FG, GE of the one parallel to the corresponding sides BC, CD, DB of the other. If BE, DG cross at A, show that CF passes through A (*i.e.* that A is an internal centre of similitude of the two similar triangles. Comp. Ex. 731).

Ex. 744.—Demonstrate VI. 32 by producing the arms of the angles GFE, BCD to meet in H, K, and showing that parallelograms EG, BD are similar. Compare Exx. 724, 725.

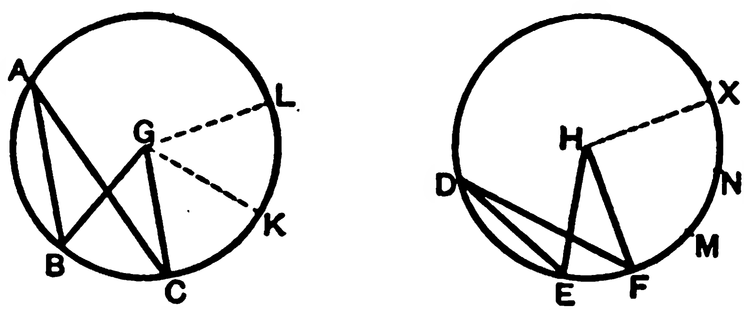
PROPOSITION 33. THEOREM.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.

Let ABC, DEF be equal ⊙s; BGC, EHF ∠s at their centres G, H; BAC, EDF ∠s at their circumferences; then

(1) arc BC : arc EF :: ∠ BGC : ∠ EHF,
:: ∠ BAC : ∠ EDF.

(2) arc BC : arc EF :: sector BGC : sector EHF.



(1) Take any number of arcs CK, KL along ○ce of ⊙ ABC, each equal to BC; and any number of arcs FM, MN, NX along ○ce of ⊙ DEF, each equal to EF. Join GK, GL, HX,

∵ arc BC = arc CK = arc KL;
∴ ∠ BGC = ∠ CGK = ∠ KGL;

∴ arc BL and ∠ BGL are equimultiples of arc BC and ∠ BGC. Similarly arc EX and ∠ EHX are equimultiples of arc EF and ∠ EHF.

Now if arc BL = arc EX,
then ∠ BGL = ∠ EHX, [III. 27.

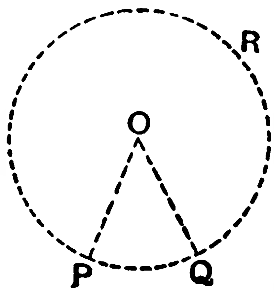
and it easily follows that

if arc BL < arc EX,
∠ BGL < ∠ EHX;
and if arc BL > arc EX,

$\angle$ BGL > $\angle$ EHX;

∴ arc BC : arc EF :: $\angle$ BGC : $\angle$ EHF, [V. Def. 5.

:: $\angle$ BAC : $\angle$ EDF.



(2) Describe a ⊙ PQR with rad. OP equal to GB or HE, and make $\angle$ POQ equal to $\angle$ BGC.

Apply ⊙ ABC to ⊙ PQR, so that G falls on O and GB along OP,

then GC falls along OQ (∵ $\angle$ BGC = $\angle$ POQ),

and C falls on Q (∵ GC = OQ),

and arc BC falls along arc PQ (∵ ⊙ ABC = ⊙ PQR);

∴ sector BGC coincides with sector POQ, and is equal to it.

Similarly each of the sectors CGK, KGL = sector POQ;

∴ sector BGC = sector CGK = sector KGL;

∴ arc BL and sector BGL are equimults. of arc BC and sector BGC.

Similarly arc EX and sector EHX are equimults. of arc EF and sector EHF.

Now if arc BL = arc. EX

then $\angle$ BGL = $\angle$ EHX,

and ∴ sector BGL = sector EHX,

and it easily follows that

if arc BL < arc EX,

sector BGL < sector EHX,

and if arc BL > arc EX,

sector BGL > sector EHX;

∴ arc BC : arc EF :: sector BGC : sector EHF.

Cor.—**In the same circle, angles, whether at the**

centre or at the circumference, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.

NOTES.

This proposition may be enunciated as follows :—

In a circle of given radius

(i) an angle at the centre **(ii) an angle at the circumference** **(iii) a sector**	**is proportional to the arc on which it stands.**

Similarly VI. 1 may be enunciated thus :—

The area of a triangle or parallelogram of given altitude is proportional to its base.

No limit is assigned in the demonstration to the magnitude of the angles BGL, EHX : they may be straight angles, or major conjugates, or even greater than four right angles.

Ex. 745.—Prove from V. Def. 5, and III. 27 that

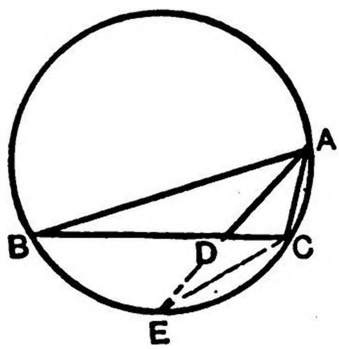
arc BC : arc EF :: $\angle$ BAC : $\angle$ EDF,

without finding the centre.

PROPOSITION B. THEOREM.

If an angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line bisecting the angle.

Let $\angle$ BAC of $\triangle$ BAC be bisected by AD, meeting BC in D; then rect. BA, AC = rect. BD, DC with sq. on AD.



Describe the circum-⊙ of $\triangle$ BAC, and produce AD to cut it at E. Join EC.

Then $\angle$ BAD = $\angle$ CAD, [Hyp.

and $\angle$ ABD = $\angle$ AEC in same segt.;

$\therefore$ $\triangle$s ABD, AEC are equiangr.; [I. 32.

$\therefore$ BA : AD :: EA : AC;

$\therefore$ rect. BA, AC = rect. EA, AD,

= rect. ED, DA with sq. on AD, [II. 3.

= rect. BD, DC with sq. on AD. [III. 35.

NOTE.

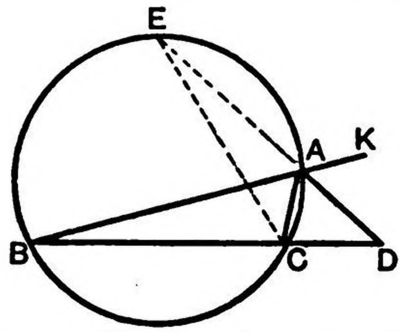
If AD were the 'external bisector' of $\angle$ BAC, with the same construction we should have

$\angle$ BAD = $\angle$ EAC,

and as in 'VI. B,

rect. BA, AC = rect. EA, AD;

∴ rect. BA, AC with sq. on AD = rect. EA, AD with sq. on AD.
= rect. ED, DA, [II. 3.
= rect. BD, DC. [III. 36, Cor.



Ex. 746.—In each figure E is the mid point of the arc BEC.

Ex. 747.—In each figure rect. ED, EA = sq. on EC.

Ex. 748.—In each figure the circle ABC is an inverse of the straight line BC with respect to E.

Ex. 749.—In each figure show that rect. AB, EC = rect. AE, BD.
Also if EB be joined, that rect. EB, AC = rect. AE, CD.

Ex. 750.—In each figure, if angle A is a right angle rect. EA, AD = twice triangle ABC.

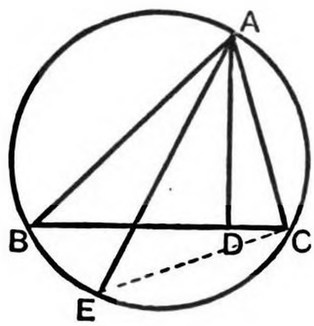
Ex. 751.—Include 'VI. B' and the extension given in the note in one enunciation.

PROPOSITION C. Theorem.

If from any angle of a triangle a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

In $\triangle$ABC let AD be drawn $\perp$r to BC, then rect. BA, AC= rect. contained by AD and the diamr. of the circum-$\odot$ of $\triangle$ABC.

Describe the $\odot$ ABC about the $\triangle$ ABC, and draw its diamr AE. Join EC.



Then rt. $\angle$ ADB= $\angle$ ACE ($\because$ AE is a diamr.),
and $\angle$ ABD= $\angle$ AEC in same segt.;
$\therefore$ $\triangle$s ABD, AEC are equiangr.;
$\therefore$ BA : AD :: EA : AC;
$\therefore$ rect. BA, AC=rect. EA, AD.

NOTE.

This proposition is often useful in affording a geometrical proof of theorems whose solution at first sight appears to require the use of Trigonometry.

It is the geometrical equivalent of a formula for the diameter (2R) of the circum-circle of a triangle, of which indeed it affords an easy proof.

It may be expressed symbolically thus—

$$2R.p_1=bc.$$

The student should notice that since $\angle$ EAC = $\angle$ BAD,

EA and AD are a pair of 'isogonal lines' (see Ex. 439); and that the property (rect. EA, AD = rect. BA, AC) is true not only for the diameter and the perpendicular, but for any other pair of 'isogonal lines.'

An instance of this occurs in the preceding proposition 'B,' where AD falls along the line EA, which is 'isogonal' with it. In the note to proposition 'B,' EA is in the same straight line with its 'isogonal.'

Ex. 752.—If, in the figure of 'VI. C,' EA cuts BC in H, and AD is produced to meet the circum-circle in K, then

rect. HA, AK = rect. BA, BC.

Ex. 753.—Any side of a triangle is to the diameter of its circum-circle as twice its area is to the rectangle contained by the other two sides.

Ex. 754.—In the fig. of VI. C, show that

rect. AB, EC = rect. AE, BD.

Ex. 755.—In the fig. of VI. C, join EB, and show that

rect. AC, EB = rect. AE, CD.

Show also that the theorem and that given in the preceding exercise are true if AD, AE are *any pair* of isogonal lines.

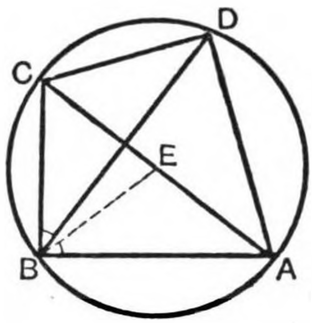
Ex. 756.—In triangle ABC, AB = AC, and D is any point in BC, or BC produced; show that the circum-circles of triangles DAB, DAC are equal.

Ex. 757.—If in the solution of the problem given as Ex. 165 (the 'ambiguous case') we obtain two triangles satisfying the given conditions, show that their circum-circles are equal.

PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.

Let ABCD be a cyclic quadl., AC and BD its diagls.; then rect. AC, BD = rect. AB, CD with rect. AD, BC.



Make ∠ ABE equal to ∠ CBD.
Then remg. ∠ CBE = remg. ∠ ABD.
and ∠ ADB = ∠ BCE in same segt.;
∴ △s ABD, CBE are equiangr.; [I. 32.
∴ AD : DB :: EC : CB;
∴ rect. AD, CB = rect. EC, DB.
Similarly rect. AB, CD = rect. AE, DB;
∴ rect. AB, CD with rect. AD, CB = rect. AE, DB with rect. EC, DB,
= rect. AC, DB.

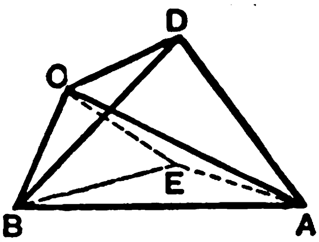
NOTE.

This proposition is sometimes called Ptolemy's Theorem. It has important applications in Geometry and Trigonometry. The student should acquaint himself with the following extension of it to quadrilaterals which are not cyclic.

If a quadrilateral cannot have a circle described about it, the rectangle contained by its diagonals is less than the sum of the rectangles contained by its opposite sides.

Let ABCD be a quadl. which is not cyclic, then rect. AC, BD $<$ rect. AB, CD with rect. AD, BC.

At B make ∠ABE equal to ∠CBD,
and ∴ remg. ∠CBE equal to ∠ABD,
and at A make ∠BAE equal to ∠BDC.



Then △s ABE, CBD are equiangr. [I. 32.
∴ BA : AE :: BD : DC ; [VI. 4.
∴ rect. AB, DC = rect. AE, BD.
Also AB : BE :: DB : BC ;
∴ AB : BD :: BE : BC, [VI. 4.
and ∠ABD = ∠CBE ; [Const.
∴ AD : DB :: EC : CB ; [VI. 6.
∴ rect. AD, BC = rect. EC, BD.
But rect. AB, DC = rect. AE, BD ;
∴ rect. AB, DC with rect. AD, BC = rect. AE, BD, with rect. EC, BD,
= rect. contd. by BD and the sum of AE, EC. [II. 1.

But since ABCD is not cyclic
∠BAC is not equal to ∠BDC ;
∴ E is not on AC ;
∴ sum of AE, EC $>$ AC ;
∴ rect. AB, DC with rect. AD, BC $>$ rect. AC, BD.

It is not difficult to adapt this demonstration so as to apply at once to Ptolemy's theorem and its extension, and to show at once that

The rectangle contained by the diagonals of a quadrilateral is equal to or less than the sum of the rectangles contained by its opposite sides according as the quadrilateral is cyclic or not.

We leave this as an exercise to the student.

Ex. 758.—**If the rectangle contained by the diagonals of a quadri-**

lateral is equal to the sum of the rectangles contained by its opposite sides, a circle can be described about it.

Ex. 759.—Enunciate the contrapositive of Ptolemy's theorem (see p. 171).

Ex. 760.—In the fig. of VI. D, show that rect. EB, BD equals rect. AB, BC.

Ex. 761.—In the fig. of VI. B, show that the ratio of EA to the sum of BA, AC is independent of the position of A on the arc BAC. See Ex. 749.

Ex. 762.—An equilateral triangle BEC is inscribed in a circle ABC. A is any point on the minor arc BC. Show *by Ptolemy's theorem* that EA=the sum of BA, AC. See Ex. 521.

DEFINITIONS.

Book VI.

1. 'Similar' rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

2. 'Reciprocal' figures, viz. triangles and parallelograms, are such as have their sides about two of their angles proportional in such a manner that a side of the first figure is to a side of the other as the remaining side of this other is to the remaining side of the first.

3. A straight line is said to be cut in 'extreme and mean ratio' when the whole is to the greater segment as the greater segment is to the less.

4. The 'altitude' of any figure is the straight line drawn from its vertex perpendicular to the base.

In the exercises and addenda which follow, the word 'segment' will frequently be used in the sense assigned to it by the following definition from the Syllabus :—

A point in a straight line is said to divide it 'internally,' and a point in the line produced is said to divide it 'externally.' In either case the distances of the point from the ends of the line are called 'segments of the line.'

MISCELLANEOUS EXERCISES.

Book VI.

Ex. 763.—S, s are the circum-centres of the similar triangles ABC, abc. Show that triangles BSC, CSA, ASB are similar respectively to triangles bsc, csa, asb.

Ex. 764.—T, t are the ortho-centres of the similar triangles ABC, abc. Show that triangles BTC, CTA, ATB are similar respectively to triangles btc, cta, atb.

Enunciate and prove similar theorems for the centroid, the mid-centre, and the symmedian point.

Ex. 765.—From the vertices A, a of the similar triangles ABC, abc, perpendiculars AD, ad are drawn to the opposite sides. Show that

$$AD : ad :: BC : bc.$$

Ex. 766.—The circum-radius and in-radius of any triangle have the same ratio as the circum-radius and in-radius of any triangle similar to it.

Extend the theorem to other pairs of corresponding lines.

In the next six exercises, taken from Euclid's *Data*, the following definitions will be required :—

Spaces, lines, and angles are said to be 'given in magnitude' when equals to them can be found.

A ratio is said to be given when a ratio of a given magnitude to a given magnitude which is the same ratio with it can be found.

Ex. 767.—If a triangle has one angle given, and if the ratio of the sum of the sides containing it to the third side be also given, the triangle is given in species (see p. 363).

Ex. 768.—If from the vertex of a triangle given in species a straight line be drawn to make a given angle with the base, it shall have a given ratio to the base.

Ex. 769.—Rectilineal figures given in species may be divided into triangles which are given in species.

Ex. 770.—If two rectilineal figures given in species have a given ratio to one another, their sides shall likewise have given ratios to one another.

Ex. 771.—If a triangle have one angle given and the ratio of the rectangle of the sides which contain it to the square of the third side be given, the triangle is given in species.

Ex. 772.—If the sides about an angle of a triangle have a given ratio to one another, and if the perpendicular from that angle to the base has a given ratio to the base, the triangle is given in species.

Ex. 773.—Prove III. 35, III. 36, and III. 37 by VI. 16.

Ex. 774.—Chords AB, AC drawn from a point A on a circle ABC are produced to meet the tangent at the other end of the diameter through A in D, E. Show that triangles AED, ABC are similar.

Ex. 775.—BD is the perpendicular from B to the internal bisector of angle BAC of triangle ABC.

If BA = 3 AC, show that AD is bisected by BC.

Produce BD, AC *to meet in* E, *and show by VI.* 1 $\triangle$BDC = $\triangle$DEC = 2$\triangle$ACD = $\frac{1}{2}$ *quad.* ABDC.

Ex. 776.—O is any point within or without triangle ABC. From A, O are drawn AD, OP parallel to one another, and meeting BC in D, P. Show that $\triangle$OBC : $\triangle$ABC :: OP : AD.

Ex. 777.—D, E, F are points in the sides BC, CA, AB of triangle ABC such that AD = BE = CF. From any point O within triangle ABC, OP, OQ, OR are drawn parallel to AD, BE, CF to meet BC, CA, AB in P, Q, R. Show that OP + OQ + OR = AD. (Bland's *Geometrical Problems.*)

Use last Exercise and V. 24.

Ex. 778.—From two given points A, O are drawn any two parallel straight lines AD, OP, meeting a given straight line in D, P. Show that the ratio AD : OP is fixed.

Ex. 779.—If each side of one triangle be perpendicular to one of the sides of a second triangle, show that the two triangles are similar.

Ex. 780.—The internal bisector of angle BAC meets BC in D. Show how to construct the triangle BAC, having given the lengths of AB, AC, AD.

See Ex. 747.

Ex. 781.—Being given the base of a triangle and the position of the bisector of the vertical angle, show how to construct the triangle; and explain the case where the solution fails.

Ex. 782.—In the fig. of 'VI. B,' show that the square on AD is a mean proportional between the difference of the squares on AB, BD and the difference of the squares on AC, CD.

Show that $BA^2 - BD^2 : AD^2 :: EA^2 - EC^2 : AC^2$,
$:: BA : AC$.
or use $BA.AC - BD.DC = AD^2$.

Ex. 783.—Adapt the enunciation of the last exercise to the external bisector, and prove the theorem.

Ex. 784.—D is a given point in a given straight line BC; it is required to find a point A such that AB, AC may be together equal to a given straight line, and such that AD bisects angle BAC.

Ex. 785.—Straight lines AOB, COD intersect in O so that AO : OB :: CO : OD. If P, Q are mid points of AB, CD, show that PQ is parallel to AC and BD.

Ex. 786.—An angle A of given magnitude subtends the chords BC in a circle ABC. Show that the ratio chord BC : diameter of circle ABC is independent of the size of the circle ABC.

Ex. 787.—In triangle ABC angle B = angle C. On BC as base construct a triangle similar to triangle ABC such that its vertical angle = either B or C.

Ex. 788.—Straight lines AOD, BOE cut at O, making AO = 2 OD and BO = 2 OE. If AE, BD are produced to meet in C, show that AC, BC are bisected at E, D.

Ex. 789.—The perpendicular bisector of one of the equal sides AC of an isosceles triangle meets the base AB in D. Show that AC is a mean proportional between AB, AD.

Ex. 790.—From any point within a regular nonagon is let fall a perpendicular on each of the sides. Show that the sum of these perpendiculars is equal to the in-radius of an nonagon, each of whose sides is equal to the perimeter of the first.

Ex. 791.—D is any point in the base AC of an isosceles triangle ABC; DE, DF are straight lines, making equal angles with AC, and meeting BC and AB in E, F. Show by VI. 15 that triangles AED, CDF are equivalent.

Ex. 792.—ABC is a triangle having angle B right; BD is drawn perpendicular to AC, and produced to E, so that DE is a third proportional to BD and DC. Show that triangles ADE, BDC are equivalent.

Ex. 793.—Two circles cut one another orthogonally (see Ex. 307). O is the centre of one of them and X a point on its circumference. Through O a straight line is drawn, cutting the other circle in P, Q; and XP, XQ cut that other circle in Y, Z. Show YZ is parallel to OX.

Ex. 794.—Find a point C in a given arc ACB of a circle such that the chord AC is double of the chord CB.

Ex. 795.—A series of circles have their centres on a given straight line and their radii proportional to their distances of the corresponding centres from a given point in that line. Find the envelope of the circles.

Ex. 796.—The in-radius of an isosceles triangle has the same ratio to one of the two equal ex-radii as the base of the triangle has to its perimeter.

Ex. 797.—In triangle ABC, AB = AC, and the circle with centre B and radius BC cuts AC in D. Show that BC is a mean proportional between AC, CD.

Ex. 798.—If D and E are points on the base BC of a triangle ABC such that AB, AC are respectively mean proportionals between BC, BD and BC, CE, show that AD, AE are each a mean proportional between BD and CE.

Ex. 799.—D, E, F are the mid points of the sides BC, CA, AB of triangle ABC. Show that a triangle can be made with its sides equal and parallel to AD, BE, CF.

Ex. 800.—A and B are two points external to a given circle DEF, and C is a point on AB such that the tangent from A to circle DEF is a mean proportional between AB, AC. If CD touches circle DEF, and AD, BF intersect on the circle at E, show that DF is parallel to AB.

Ex. 801.—The diagonals of the quadrilateral ABCD are perpendicular to one another, and E, F are the projns. of C on AB, AD. Show that triangles AEF, ADB are similar.

Ex. 802.—ABCD is a square, P any point on the minor arc AB of its circum-circle. Show that rect. PC, PD = rect. PA, PB + rect. PB, PC + rect. PD, PA.

Show that $\triangle$ PCD − $\triangle$ PAB = $\triangle$ PBC + $\triangle$ PDA, *and then that* $\triangle$*s are in same ratio as corresponding rectangles.*

Ex. 803.—If two circles are drawn, one touching the sides of an equilateral triangle at the ends of the base, the other cutting the first orthogonally at the same points, either common tangent is equal to the chord of contact of the first circle, and is a mean proportional between its diameter and a side of the triangle.

Ex. 804.—The Simson lines of a triangle with respect to the ends of a diameter are perpendicular to one another.

Ex. 805.—The Simson lines of any two points P and Q on the circum-circle of triangle ABC are inclined to one another at an angle equal to that which PQ subtends at the circumference.

Ex. 806.—If three straight lines be proportionals representing the first three terms of a decreasing series, find by a geometrical construction a straight line equal to the sum of the series *ad infinitum.*

Use V. 12.

Ex. 807.—Similar triangles are to another as the squares of their (i) circum-radii, (ii) in-radii, (iii) corresponding ex-radii, (iv) corresponding altitudes.

Ex. 808.—In triangle ABO angle BAC is obtuse; AD, AE are drawn to meet BC in D, E, so that angle ADB = angle BAC = angle AEC. Show that triangles ABD, AEC, ABC are to one another as the squares on the sides of triangle ABC.

Ex. 809.—Make a square that shall bear a given ratio to a given square.

Ex. 810.—Make a rectilineal figure similar to a given rectilineal figure and bearing a given ratio to another given rectilineal figure.

Ex. 811.—If in the fig. of IV. 10, AE and BD are produced to meet in F, show that BF is divided at D in extreme and mean ratio.

Show also that the internal bisector of angle BAD divides CD in extreme and mean ratio.

Ex. 812.—ABCD is a parallelogram; E a point in AB such that AE is less than EB; CF, equal to AE, is drawn perpendicular to DC on side remote from AB; and FG, equal to EB, is drawn to meet DC produced in G. If GE, DA meet when produced in H, show that triangle HDG = parallelogram ABCD equal to angle PAQ.

Ex. 813.—E, F, G, H are points in the sides AB, BC, CD, DA of a square ABCD such that AE = $\frac{1}{2}$ AB, BF = $\frac{1}{3}$ BC, CG = $\frac{1}{4}$ CD, DH = $\frac{1}{6}$ DA. Show that quadrilateral EFGH is equivalent to the square on GD.

Ex. 814.—The sides of a cyclic quadrilateral taken in order are in proportion. Show that one of its diagonals bisects the other.

Ex. 815.—If one diagonal of a cyclic quadrilateral bisects the other, show that the sides taken in order are in proportion.

Ex. 816.—Deduce III. 4 from Ex. 815.

Ex. 817.—The equal sides AB, AC of an isosceles triangle are produced to E, F, so rect. BE, CF = square on AB. Show that EF passes through a fixed point.

Complete the parallelogram ABDC, *and show by VI.* 32 *that* ED, DF *are in the same straight line.*

Also prove without the use of VI. 32.

Ex. 818.—If in the fig. of 'VI. C' the tangent at A be drawn to meet BC produced in F, the diameters of the circum-circles of triangles ABF, ACF are in the ratio AF : CF.

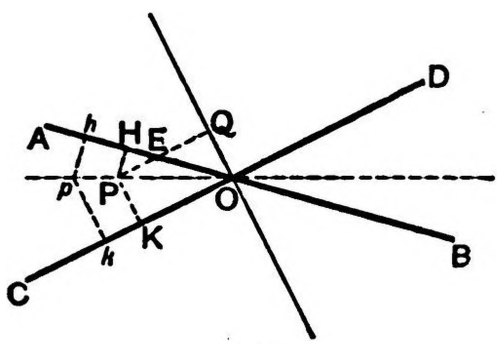
Ex. 819.—If in the figure of 'VI. C' AD is produced to cut the circle in F and AH drawn parallel to CF to meet BC, or BC produced in H, show that rect. AH, FB = rect. BA, AC.

ON LOCI, 2. *See p.* 107.

PROPOSITION 1.

The locus of a point whose distances from two given intersecting straight lines are in a constant ratio is a pair of straight lines passing through the cross of the two given straight lines.

Let AB, CD be the two given st. lines intersecting at O, and let the given ratio be that of the given st. line X to the given straight line Y; then the locus of a point whose distances from AB and CD are in the ratio X : Y is a pair of st. lines through O.



Find points P, Q within ∠s AOC, AOD, whose distances from AB, CD = X, Y (see p. 109).

Join PO, QO, and produce the joining lines indefinitely: these form the locus.

(i) Let p be any pt. on the st. line through P, O; draw ph, PH ⊥r to AB; pk, PK ⊥r to CD.

Then △s Ohp, OHP are equiangr.;

∴ ph : pO :: PH : PO.

Similarly pO : pk :: PO : PK;

∴ ph : pk :: PH : PK. [Ex æquali.

:: X : Y,

i.e. distances from p to AB, CD are in ratio X : Y.

Similarly distances from any point in st. line through Q, O are in ratio X : Y.

(ii) Let distances ph, pk of any pt. p within ∠s AOC, BOD to AB, CD be in ratio X : Y.

Join HK, hk.

In quadls. pkoh, PKOH,

∠s pko, koh, ohp = ∠s PKO, KOH, OHP;

∴ fourth ∠ hpk = fourth ∠ HPK. [I. 32, Cor.

But hp : pk :: X : Y,

:: HP : PK;

∴ ph : hk :: PH : HK,
and ∠ phk = ∠ PHK. } [VI. 6.

But rt. ∠ phO = rt. ∠ PHO;

∴ ∠ khO = ∠ KHO;

∴ △s khO, KHO are equiangr.;

∴ hk : hO :: HK : HO.

But ph : hk :: PH : HK;

∴ ph : hO :: PH : HO.

But rt. ∠ phO = rt. ∠ PHO;

∴ ∠ pOh = ∠ POH;

∴ p is on the line through P, O.

Similarly if the distance of any pt. within the ∠s AOD, BOC are in the ratio X : Y, the pt. lies on the line through Q, O.

NOTE.

The theorem is a generalisation of 4, p. 107 (see diagram on p. 51).

The demonstration applies equally well to the case in which ph, pk, instead of being drawn *perpendicular* to AB, CD, are drawn *in any given directions*, e.g. *parallel to two given straight lines.*

A limiting case of the general proposition thus indicated is that in which ph, pk are *in the same straight line drawn in a given direction*, of which we give a separate demonstration below.

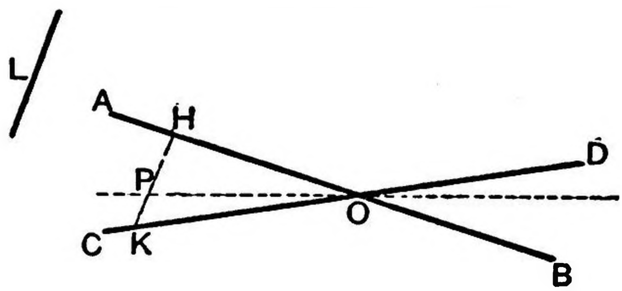
PROPOSITION 2.

The locus of a point which divides into segments having a given ratio to one another the intercept made by two given intersecting straight lines on a parallel to a third given straight line is *in general* a pair of straight lines passing through the cross of the two given intersecting straight lines.

Let HK be an intercept made by the two given st. lines AB, CD crossing at O on a parallel to a given st. line L, and P a point dividing HK internally; then

(i) If P lies on a fixed st. line through O,
HP : PK in a constant ratio.

(ii) If HP : PK in a constant ratio,
P lies on a fixed st. line through O.



(i) Let P lie on a fixed st. line through O,
∵ ∠s of △ OHP are constant;
∴ ratio HP : OP is constant. [VI. 4.
Similarly ratio OP : PK is constant;
∴ ratio HP : PK is constant.

(ii) Let ratio HP : PK be constant,
∵ ratio HP : PK is constant;
∴ ratio HP : HK is constant.
But ∠s of △ OHK are constant,
∴ ratio HK : OH is constant;
∴ ratio HP : OH is constant;

but $\angle$ OHP is constant;
$\therefore$ $\angle$ HOP is constant; [VI. 6.
$\therefore$ P lies on a fixed st. line through O.

Similarly for a point of external division of HK; and as for all ratios *except that of equality* one external as well as one internal pt. can be found (see p. 274), the locus in general consists of a pair of straight lines. See Ex. 109.

Contrast the method of proof used for Prop. 2 with that used for Prop. 1.

It would be a useful exercise for the student to demonstrate Prop. 1 by the method used for Prop. 2, and *vice versâ*.

The method of Prop. 2 is frequently adopted to save the construction of a complicated figure.

Ex. 820.—To find a point whose distances from the three sides of a given triangle are to one another as three given straight lines.

Note that there is only one such point within the triangle, and there are three without. Compare p. 107, *Intersection of Loci*, 2.

Ex. 821.—A circle cuts the side BC of a triangle ABC in α, λ; CA in β, μ, and AB in γ, ν. Show that if $\beta\gamma$, $\mu\nu$ intersect in P, $\gamma\alpha$, $\nu\lambda$ in Q; and $\alpha\beta$, $\lambda\mu$ in R; then AP, BQ, CR are concurrent.

They meet at a point whose distances from the sides are proportional to $\alpha\lambda$, $\beta\mu$, $\gamma\nu$. *Show ratio of perpendiculars from* P *by similarity of triangles* P$\beta\mu$, P$\gamma\nu$.

PROPOSITION 3.

The locus of a point which divides into segments having a given ratio to one another the intercept made by two given parallel straight lines on any third straight line is a pair of straight lines parallel to the two given ones.

Let AB and CD be the two given $\parallel$ st. lines; HK the intercept made by them on any third st. line.

P any point on HK or HK produced.

Show (*i*) *that if* P *lies on a* || *to* AB, CD, *the ratio* HP : PK *remains constant;*

(*ii*) *that if the ratio* HP : PK *remains constant*, P *lies on one of two* ||*s to* AB, CD.

For the method of proof, see VI. 10, *where* FD, BC *represent the two given* || *straight lines*, BF *and* CD *the intercepts on any other two straight lines.*

PROPOSITION 4.

O is a given point in a given straight line AB; E any other point in AB. Show that the locus of the end of a straight line drawn from E in a given direction, and bearing a constant ratio to OE, is a pair of straight lines through O.

Let the given direction be that of the st. line COD through O, and let EP be || to CD on the same side of AB as C is. See p. 427.

First let P lie on a st. line through O,

then ∠s of △ OEP are constant;

∴ OE : EP is constant.

Next let OE : EP be constant;

then ∵ ∠ OEP is also constant,

∴ ∠ AOP is constant;

∴ P lies on a fixed st. line through O.

Similarly when P is on the opposite side of AB.

Ex. 822.—AB, DC are parallel sides of a trapezoid ABCD; PQ any points on AB, DC, such that PQ passes through a given point O on the join EF of the mid points E, F of AD, BC. Show that

APQD : PBCQ :: EO : OF.

First, let PQ, AB, DC *be concurrent, and then remove the restriction.* Solve also when the trapezoid is a parallelogram.

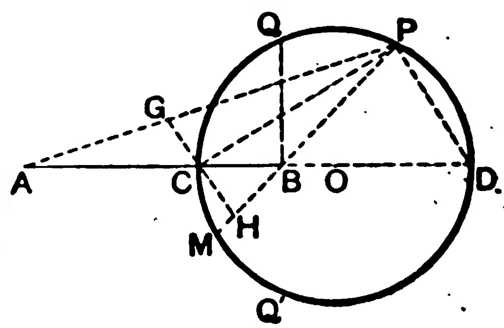
PROPOSITION 5.

The locus of a point whose distances from two given points are in a constant ratio of inequality is a circle.

Let A and B be the two given pts.; C, D the pts. which divide AB internally and externally into segments which have to one another the given ratio of inequality; then

(i) if a pt. P be such that AP : PB in the given ratio, P lies on the ⊙ described on CD as diamr.

(ii) if P lies on the ⊙ described on CD as diamr. AP : PB in the given ratio.



(i) Let AP : PB in the ratio.

Join CP, PD.

∵ AP : PB :: AC : CB;

∴ CP is the intl. bisector of ∠APB. [VI. 3.

Similarly DP is the extl. bisector; [VI. A.

∴ ∠CPD is a rt. ∠;

∴ P lies on the ⊙ described on CD as diamr.

(ii) Let P be any pt. on the ⊙ described on CD as diamr. Join CP, PD.

Through C draw GCH ∥ to PD, meeting AP, BP in G, H.

Then $\triangle$s ACG, ADP are equiangr.;

$\therefore$ CG : AC :: DP : AD;

but AC : CB :: AD : DB;

$\therefore$ CG : CB :: DP : DB,

:: CH : CB ($\because$ $\triangle$s DPB, CHB are equiangr.);

$\therefore$ CG = CH.

But GH is $\perp$r to CP;

$\therefore$ CP bisects $\angle$ APB;

$\therefore$ AP : PB :: AC : CB.

Ex. 823.—A, B, C, D are four points in a straight line ABCD.

Find a point P such that angle APB = angle BPC = angle CPD. A solution of this problem appeared in an article in the *Acta Eruditorum* for 1702, reviewing a work by Ceva. It seems to have been previously given in Ozanam's *Mathematical Dictionary* (1690).

Ex. 824.—Find a point whose distances from three given points shall be to one another as three given straight lines.

Ex. 825.—Find the locus of a point such that if a pair of tangents be drawn from it to each of two given circles, the angles contained by each pair shall be equal.

Ex. 826.—Find a point such that if a pair of tangents be drawn from it to each of three given circles, the angles contained by each pair shall be equal.

PROPOSITION 6.

If O is a given point, and P any point on a given straight line, the locus of a point p, such that Op falls along OP and bears a given ratio to it, is a parallel to the given straight line.

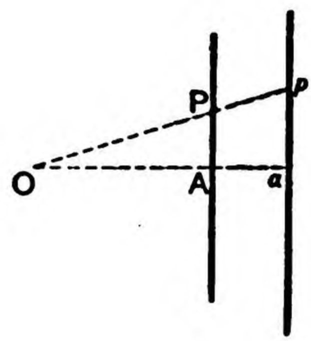
Draw OA $\perp$r to the given st. line, and on OA, produced if necessary, take a such that Oa : OA in the given ratio; then

(i) if OP, or OP produced, meets the ∥ through a to the given st. line in p, Op : OP in the given ratio.

(For Op : OP :: Oa : OA.)

(ii) If p be taken on OP, or OP produced, such that Op : OP in the given ratio, ap is ∥ to AP.

(For Op : OP :: Oa : OA.)



The demonstration might easily be adapted to the case in which Op falls in the same straight line with OP on the opposite side of O, *i.e.* when Op makes a 'straight angle' with OP. This is a limiting case of Prop. 7.

PROPOSITION 7.

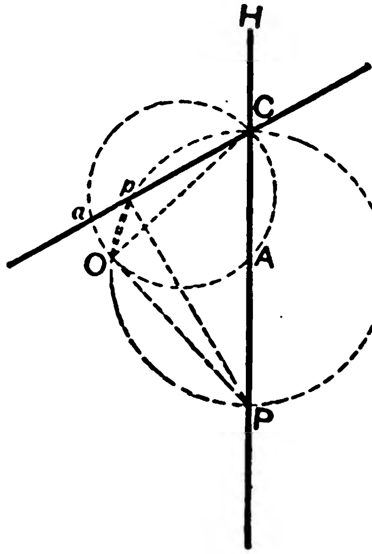
If O is a given point, and P any point on a given straight line, the locus of a point p, such that Op makes a given angle in a given sense with OP and bears a given ratio to it, is a straight line.

Let the circum-⊙ of △ OpP cut the given st. line PH again at C.

Then ∵ Op : OP in a constant ratio,

and ∠ POp is constant,

∴ ∠ OpP is constant;
∴ ∠ OCP in the same segment is constant;
∴ C is a fixed point.



Also ext. ∠ pCH of cyclic quadl. OpCp= ∠ POp, and is ∴ constant;

∴ p lies on a fixed st. line through C.

The demonstration of the converse is left to the student.

PROPOSITION 8.

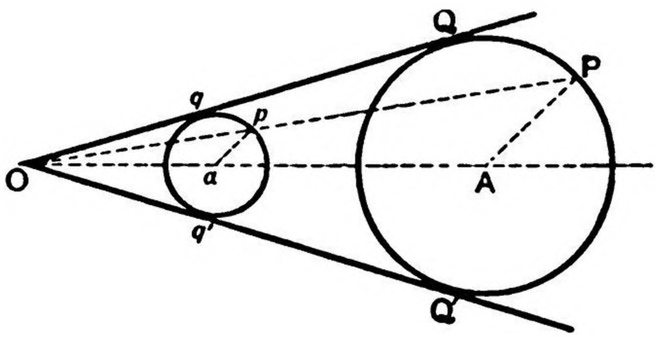
If O is a given point and P any point on a given circle, the locus of a point p, such that Op falls along OP and bears a given ratio to it, is a circle.

Let A be the centre of the given ⊙, and let a be taken on OA, or OA produced, such that Oa : OA in the given ratio. Join ap.

Then Op : OP :: Oa : OA;

∴ Op : Oa :: OP : OA;

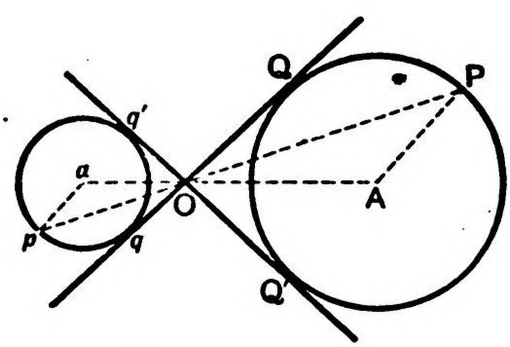
and ∠ AOP is common to △s aOp, AOP;
∴ ap : Oa :: AP : OA.



But Oa, AP, and OA are constant;
∴ ap is constant;
∴ p lies on a ⊙ whose centre is a.

The demonstration of the converse is left to the student.

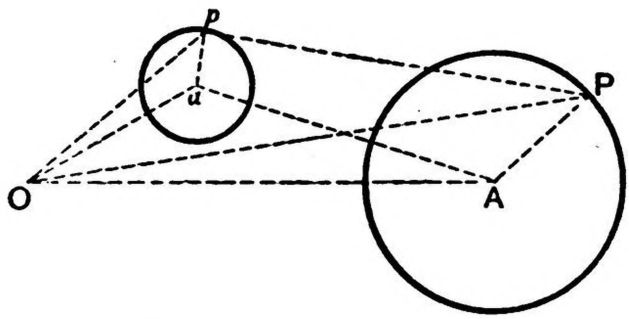
The demonstration might easily be adapted to the case in which Op falls in the same st. line, with OP on the opposite side of O; *i.e.* when Op makes a 'straight angle' with OP. This is a limiting case of Prop. 9. On account of its importance we add a figure illustrating this case, leaving the demonstration to the student.



Ex. 827.—From two given points O, o two parallel straight lines OP, op are drawn, having a given ratio to one another: P lies on a given circle. Find the locus of p.

PROPOSITION 9.

If O is a given point and P any point on a given circle, the locus of a point p, such that Op makes a given angle in a given sense with OP, and bears a fixed ratio to it, is a circle.



Let A be the centre of the given circle, and let Oa be taken, making the given angle in the given sense with OA, and bearing the given ratio to it. Join ap.

$\because \angle AOa = \angle POp$;

$\therefore \angle aOp = \angle AOP$.

Again $\because Op : OP :: Oa : OA$,

$\therefore Op : Oa :: OP : OA$;

but $\angle aOp = \angle AOP$;

$\therefore ap : Oa :: AP : OA$.

But Oa, AP, OA are constant;

$\therefore$ ap is constant;

$\therefore$ p lies on a ⊙ whose centre is a.

Ex. 828.—From a given external point O, any straight line OPQ is drawn, cutting a given circle in P, Q, and on it is taken a point R such that OR bears a fixed ratio to the sum of OP, OQ. Show that the locus of R is a circle through O.

A hint may be found in Ex. 359.

Investigate a similar theorem for an internal point.

Ex. 829.—P is any point on a given arc APB; O the mid point of the conjugate arc AOB. If Q be taken on OP such that OQ bears a constant ratio to the sum of AP, PB, find the locus of Q.

Ex. 830.—To draw a triangle of given species which shall have one vertex at a given point and the other two vertices on given straight lines or given circles.

Compare Ex. 157.

Ex. 831.—Show that any number of triangles of given species can be described having one vertex on each of three given straight lines or circles.

Compare Ex. 157.

Ex. 832.—From a given point O two straight lines OP, OQ are drawn, such that angle POQ and rect. OP, OQ are constant. P lies (1) on a given straight line, (2) on a given circle. Find the locus of Q.

See pp. 251-254.

HARMONIC DIVISION.

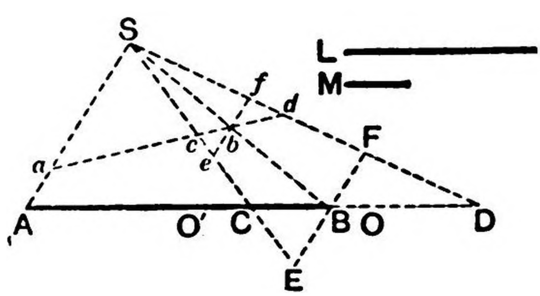
DEF.—**A point in a straight line is said to divide it 'internally,' and a point in the line produced is said to divide it 'externally.' In either case the distances of the point from the ends of the line are called 'segments' of the line.**

PROPOSITION 1.

To find the points which divide a given straight line into segments which have a given ratio to one another.

Let AB be the given st. line, and let the given ratio be that of the st. line L to the st. line M.

From A in any direction draw AS equal to L, and through B draw EBF ∥ to AS, such that EB=BF=M. Join SE, SF. Let SE cut AB in C, and SF, AB produced cut in D, then C and D are the reqd. pts.



For AC : CB :: AS : EB (∵ ∠s CEB, CSA are equiangr.),

:: L : M,

:: AS : BF,

:: AD : DB (∵ ∠s DFB, DSA are equiangr.).

Def.—When the internal segments AC, CB of a given straight line AB have the same ratio to one another as the external segments AD, DB,

(i) the straight line AB is said to be 'harmonically divided at the points C, D;

(ii) the points A, C, B, D are said to form a 'harmonic range';

(iii) AC, AB, AD are said to be in 'harmonic progression.'

(iv) AB is called the 'harmonic mean' between AC, AD;

(v) the points C and D are called 'harmonic conjugates' with respect to the points A and B.

PROPOSITION 2.

If C, D are harmonic conjugates, with respect to A, B, then A, B are harmonic conjugates with respect to C, D.

For DA : DB :: AC : BC;
∴ DA : AC :: DB : BC,
i.e. D, B, C, A form a harmonic range.

PROPOSITION 3.

If three points A, B, D are given in a straight line, one point C, and one only, can be found forming a harmonic range A, C, B, D with A, B, D.

Take any pt. S outside the given st. line, and join SA, SB, SD. Through B draw FBE ∥ to SA, meeting SD in F, and making BE = FB. Join SE, cutting AB in C. Then by the demonstration of Prop. 1, A, C, B, D is a harmonic range.

Again, if A, C, B, D, forming a harmonic range, be joined with an external point S, and EBF be drawn || to AS and meeting SC, SD produced in E, F,

AS : EB :: AC : CB ($\because$ $\triangle$s CEB, CSA are equiangr.),
:: AD : DB,
:: AS : BF;
$\therefore$ EB = BF.

Now when A, B, D are given and S taken, only one line through S can be drawn, making EB = BF.

$\therefore$ only one point C can be found making a harmonic range A, C, B, D with A, B, D.

PROPOSITION 4.

If A, C, B, D form a harmonic range, AB(AC + AD) = 2 AC.AD.

For AB(AC + AD) = AB.AC + AB.AD, [II. 1.
= AB.AC + CB.AD + AC.AD, [II. 1.
= AB.AC + DB.AC + AC.AD ($\because$ CB.AD = DB.AC)
= 2 AC.AD.

Cor.—If O be the mid point of CD,
AB.AO = AC.AD ($\because$ AC + AD = 2 AO).

Also since D, B, C, A form a harmonic range,
DC(DB + DA) = 2 DB.DA;
and if O′ be the mid point of AB,
DC.DO′ = DB.DA.

PROPOSITION 5.

If A, C, B, D form a harmonic range and O, O′ are the mid points of CD, AB,

$$OC^2 = OD^2 = OA.OB,$$
$$A'O^2 = O'B^2 = O'C.O'D.$$

For DA : AC :: DB : BC;

$\therefore$ DA+AC : DA−AC :: DB+BC : DB−BC;

$\therefore$ 2 OA : 2 OC :: 2 OC : 2 OB;

$\therefore$ OC^2=OA.OB.

Similarly $O'A'^2$=O'C.O'D.

PROPOSITION 6.

If S be any point without the straight line in which is the harmonic range ACBD, and if through B a straight line EF be drawn parallel to SA and meeting SC, SD in E and F, then EB=BF.

See demonstration of 2d part of Prop. 3.

The converse theorem has been already demonstrated. (See Prop. 1.)

PROPOSITION 7.

If four straight lines SA, SC, SB, SD, drawn from a point S, cut any transversal ACBD in points A, C, B, D, forming a harmonic range, they will cut any other transversal acbd in points a, c, b, d, forming a harmonic range.

For through B, b draw EBF, ebf ∥ to AS, and meeting SC in E, e and SD in F, f.

Then EB=BF; [Prop. 6.

$\therefore$ eb=bf;

$\therefore$ a, c, b, d is a harmonic range. [Prop. 1.

Def.—A system of four straight lines SA, SB, SC, SD, drawn from a point S such that the four points a, c, b, d, in which any transversal cuts them, form a harmonic range, is called a 'harmonic pencil'; the straight lines SC, SD passing through the conjugate points c, d are called 'conjugate rays'; so also are the straight lines SA, SB passing through the conjugate points a, b.

Thus in the figure on p. 360, AB, AD, AC, AE form a harmonic pencil, of which the two arms AB, AC of the ∠ BAC are one pair of conjugate rays, and the internal and external bisectors AD, AE of the ∠ BAC are the other pair.

Again, the two straight lines OP, OQ (diagm. on p. 427) form a harmonic pencil with the two given st. lines OA, OB. For PQ can easily be shown to be || to OC and bisected by OA.

Also since any antiparallel $\beta\nu$ to the base BC of △ ABC is || to the tangent at A to the circum-⊙ of ABC, it is clear that this tangent forms, with the sides AB, AC and the symmedian through A, a harmonic pencil, of which the tangent and the symmedian are one pair of conjugate rays, and the sides of the triangle the other pair.

PROPOSITION 8.

If three straight lines SA, SB, SD are given, drawn from the same points, one straight line SC, and one only, can be found forming a harmonic pencil SA, SC, SB, SD with SA, SB, SD.

Draw a transversal A, B, D; then by Prop. 3 one point C, and one only, can be found making a harmonic range A, C, B, D with A, B, D.

Hence one st. line SC, and one only, can be found, etc.

PROPOSITION 9.

If one pair of conjugate rays of a harmonic pencil are at right angles to each other, they are the external and internal bisectors of the angle between the other two.

In the diagram let ASB be a rt. ∠.

Then SB is at rt. ∠s to EF (∵ EF is || to AS);

$\therefore \angle ESB = \angle FSB$ (I. 4),
i.e. SB is the intl. bisector of $\angle CSD$;
$\therefore$ SA is the extl. bisector.

Cor.—If S is any point on the circle having AB for diameter DS : SC :: DB : BC. See Ex. 697.

Ex. 833.—A, B, C, D are four given concyclic points, and S any fifth point on the same circle. If SA, SB, SC, SD form a harmonic pencil for any one position of S, they do so for all positions.

Prove by III. 21 *and Superposition.*

Ex. 834.—A, B, C, D are four given concyclic points, and S any fifth point on the same circle. If the tangent at D form a harmonic pencil with DA, DB, DC, show that SA, SB, SC, SD also form a harmonic pencil, and that the tangent at A forms a harmonic pencil with AB, AC, AD.

Ex. 835.—A and B are two given points on a given circle, whose centre is O. It is required to find a point P on the same circle, such that PA, PB intercept equal segments OM, ON on a given diameter CD of the circle.

Rouché et de Comberousse, *Traité de Géométrie Elémentaire.*

Produce AB, CD *to meet in* T, *and draw tangents* TE, TF *to the given circle. The other end* P *of the diameter through* F *or* E *will be the point required. Remember that* EA, EF, EB, ET, *and* FA, FE, FB, FT *are harmonic pencils, and use Ex.* 834.

PROPOSITION 10.

Through a fixed point A any straight line is drawn, meeting two given intersecting straight lines SC, SD in C and D. If AB is taken along ACD, a harmonic mean between AC, AD, then B lies on a fixed line through S.

Join AS. Through B draw EBF ∥ to AS, meeting AC, AD in E, F.

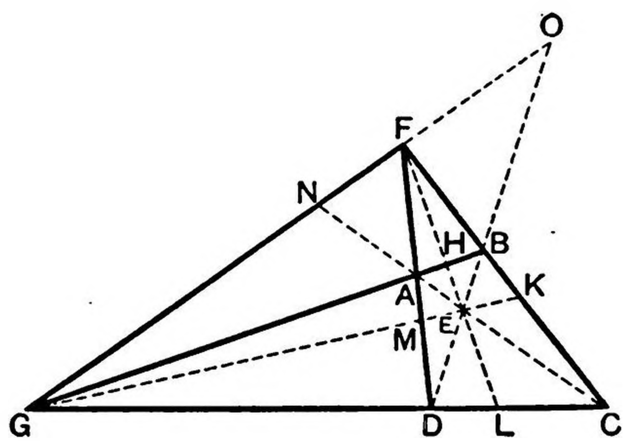
Then, as in Prop. 3, B is the mid pt. of EF.
But EF is ∥ to a fixed st. line AS;
$\therefore$ its mid pt. B lies on a fixed st. line through S (see Ex. 109).

The straight line SB is called the **harmonic polar**, or simply the **polar** of A, with respect to the two st. lines SC, SD.

PROPOSITION 11.

If the diagonals AC, BD of a quadrilateral ABCD cross at E; the sides DA, CB produced meet in F; and the sides BA, CD produced in G;

then (i) FG, FA, FE, FB
(ii) GF, GA, GE, GD
(iii) EG, EA, EF, EB
form a harmonic pencil.



Along GAB, GDC, take GH, GL, the harmonic means between GA, GB and GD, GC.

Then H, L lie on a st. line through F, and also on a st. line through E; [Prop. 9.

∴ they lie on the st. line FE;

∴ FG, FA, FE, FB form a harmonic pencil.

Similarly GF, GA, GE, GD form a harmonic pencil.

Also since G, A, H, B form a harmonic range,

∴ EG, EA, EH, EB form a harmonic pencil.

Cor. i.—With the same construction, let the diagonals AC, BD be produced to meet the straight line through F and G in N, O; then G, N, F, O is a harmonic range.

Ex. 836.—**To find the polar of a given point F with respect to two given straight lines by means of the ruler only.**

Draw FAB, FBC, *cutting the two given straight lines in* A, B, D, C. (*See fig. of Prop.* 10.) *Join* AC, BD, *crossing at* E: *then* E *is a point on the required polar. Similarly another point can be found.*

Ex. 837.—**Through a given point E to draw a straight line, such that it would, if produced, pass through the point of intersection G of two given straight lines when that point is inaccessible.**

Through E *draw* AEC, BED, *meeting the given straight lines in* A, B, D, C (*fig. of Prop.* 11), *and produce* DA, BC *to meet in* F. *Through* F *draw any other straight line* Fbc, *cutting* AB *in* b *and* CD *in* d; *and let* bD, cA *cross at* e. *Then* Ee *is the required line.*

Compare Ex. 241.

Ex. 838.—AB, CD are two given parallel straight lines. E is a given point. It is required to determine, by means of a ruler only, a straight line through E parallel to AB, CD.

Ex. 839.—AB and CD are two given parallel straight lines. It is required to bisect a finite portion of either of them by means of a ruler only.

Ex. 840.—ABCD is a given parallelogram, and EF a given finite straight line. Show how to bisect EF by means of a ruler only.

Construct by means of Ex. 838, *a parallelogram having* EF *for one diagonal.*

PROPOSITION 12.

From a given external point A, a straight line APQ is drawn, cutting a given circle in P and Q; along APQ is taken the harmonic mean AR between AP and AQ; then the locus of R is the polar of A with respect to the circle.

Find the centre O of the given ⊙, and let the straight line through A, O cut the ⊙ in C and D. Bisect PQ in E, and join OE. Along ACD take AB the harmonic mean between AC, AD.

Then AB.AO=AC.AD,	[Prop. 4.
=AP.AQ,	[III. 36, Cor.
=AR.AE;	[Prop. 4.

∴ B, O, E, R are concyclic;
∴ ∠CBR = ∠OER
which is a rt. ∠;
but B is a fixed point;
∴ R lies on a fixed st. line.
Also since $OA.OB = OC^2$, [PROP. 5.
this st. line is the polar of A with respect to the given ⊙. (See pp. 243, 244.)

Hence this straight line is sometimes called the '**harmonic polar**' of A with respect to the given ⊙. (Compare Prop. 10.)

If in the figure of Prop. 11 the quadrilateral ABCD is *cyclic*, and H and L be taken as before, then H, L lie on the polar of G with respect to the ⊙ round ABCD.

Hence this polar passes through E, F.

Hence we are enabled to solve the following problem :—

PROPOSITION 13.

To draw a pair of tangents to a given circle from a given external point by means of a ruler only.

From the given extl. pt. G draw GAB, GCD cutting the given ⊙ in A, B and C, D. Join AC, BD, crossing at E. Join DA, CB, and produce them to meet at F; then EF will cut the ⊙ at the points of contact of the reqd. tangents from G.

Ex. 841.—A, B, C, D are four given concyclic points, and P any fifth point on the same circle. The tangents at A, B, C, D cut the tangent at P in a, b, c, d. If abcd is a harmonic range for one position of P, it is so for all positions.

Show that if abcd is a harmonic range, PA, PB, PC, PD form a harmonic pencil, and conversely, and use Ex. 833.

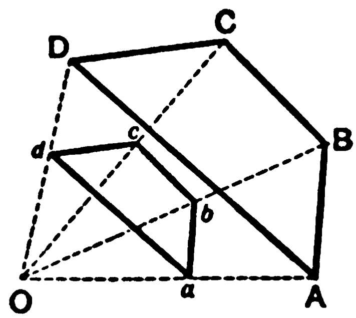
ON SIMILARITY.

PROPOSITION 1.

If along the straight lines OA, OB, OC, OD, joining a given point O to the corners A, B, C, D of a given rectilineal figure ABCD, a, b, c, d, be taken, such that

Oa : OA = Ob : OB = Oc : OC = Od : OD,

then a, b, c, d are the corners of a rectilineal figure abcd similar to ABCD.



Join ab, bc, cd, da,
∵ Oa : OA :: Ob : OB;
∴ ab is ∥ to AB;
∴ ∠Oab = ∠OAB,
and Oa : ab :: OA : AB. [VI. 4.
Similarly ∠Oad = ∠OAD,
and ad : Oa :: AD : OA;
∴ remg. ∠dab = remg. ∠DAB,
and da : ab :: DA : AB. [Ex æquali.

Similarly ∠s abc, bcd, cda = ∠s ABC, BCD, CDA, and the sides about them are propls.;

∴ abcd is similar to ABCD.

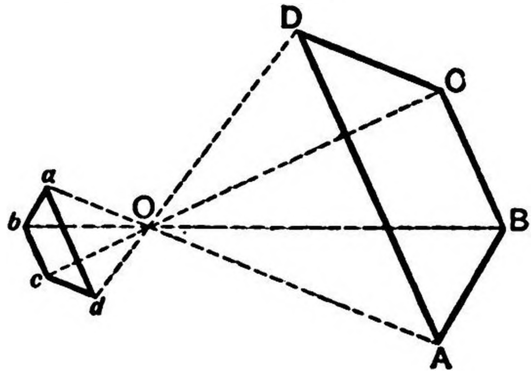
Def.—The point O is called the **external centre of similitude** of the similar figures abcd, ABCD.

PROPOSITION 2.

If the straight lines AO, BO, CO, DO, joining the corners A, B, C, D of a given rectilineal figure ABCD to a point O be produced to points a, b, c, d, such that

$$Oa : OA = Ob : OB = Oc : OC = Od : OD,$$

then a, b, c, d are the corners of a rectilineal figure abcd similar to ABCD.



The demonstration resembles that of Prop. 1.

DEF.—O is called the **internal centre of similitude** of the two similar figures abcd, ABCD.

PROPOSITION 3.

Two given similar figures can always be placed so that the lines joining corresponding points are concurrent, *i.e.* so as to have a centre of similitude.

Let abcd be placed so that ab is $\parallel$ to AB and $\angle$ abc in the same sense as the $\angle$ ABC, and let Aa, Bb, produced if necessary, meet in O.

Join OC, Oc.

Then $\angle$ Oba $= \angle$ OBA,

and Ob : ba :: OB : BA. [VI. 4.

But $\angle$ abc $= \angle$ ABC, [HYP.

and ab : bc :: AB : BC;
$\therefore$ $\angle$Obc $=$ $\angle$OBC,
and Ob : bc :: OB : BC; [Ex æquali.
$\therefore$ $\angle$bOc $=$ $\angle$BOC;
$\therefore$ c, O, C are in the same st. line.
Also bc is $\parallel$ to BC ($\because$ $\angle$Obc $=$ $\angle$OBC),
and $\therefore$ Oc : OC $=$ Ob : OB.
Similarly d, O, D are in the same st. line,
and Od : OD $=$ Oc : OC.

Note that when two similar rectilineal figures abcd, ABCD have a centre of similitude O, any two corresponding sides ab, AB are parallel to one another, and that the ratio ab : AB (called the 'ratio of similitude' of the two figures) is the same as that of the distances of any two corresponding points a, A from O.

If the ratio of a pair of corresponding sides (ab : AB) of two given similar rectilineal figures is not one of equality, the two figures can be placed so as to have either an external or an internal centre of similitude, according as ba is placed, as in the first or as in the second diagram.

If the ratio ab : AB is one of equality (*i.e.* if the figures abcd, ABCD are *congruent*), we cannot place the figures so as to have an external centre of similitude; for if we place ab as in the first diagram, Aa and Bb are parallel.

We can still, however, get an internal centre; for if we place ab as in the second diagram, Aa, Bb cut one another, and it is easy to show that

Oa $=$ OA, Ob $=$ OB, Oc $=$ OC, Od $=$ OD.

This special case is one of great importance. The two figures abcd, ABCD are then said to be **symmetrical with regard to the point** O, and O is called a **centre of symmetry** of the two figures.

Def.—**When two similar figures have a centre of similitude, they are called 'homothetic'; and are said to be 'similarly placed' or 'oppositely placed' according as that centre is external or internal.**

Def.—**When a figure is such that a point O exists within it such that every straight line drawn through O and terminated by the boundary of the figure is bisected at O, the figure is said to be 'symmetrical with respect to the point O,' and O is called 'the centre of symmetry' of the figure.**

Thus a parallelogram is symmetrical with respect to the cross of its diagonals, and a circle is symmetrical with respect to its centre.

Contrast '*central symmetry*' with '*axial symmetry*' (see p. 23).

A knowledge of the properties of homothetic figures can be turned to good account in Geometrical Drawing.

Suppose, for instance, we are required to draw a square bcde, such that b, c fall on the sides AB, AC of a given triangle ABC, and d, e on the base BC.

Using the 'Method of Analysis and Synthesis' (see p. 114), suppose the required square bcde were drawn, and also that a square BCDE were described on the base BC on the side remote from A.

Then bcde, BCDE would be homothetic figures, and b, c lie on AB, AC respectively: d, e must lie on AD, AE. Hence the construction:—

Construct a square BCDE on the base BC on the side remote from A. Join AD, AE, cutting BC in d, e. Through d, e draw dc, eb parallel to EB or DC, meeting AB, AC in b, c; bcde shall be the required square.

$$\text{For be : BE :: Ae : AE,}$$
$$\text{:: ed : ED;}$$
$$\therefore \text{be} = \text{ed}.$$
$$\text{Similarly cd} = \text{ed}.$$

We leave the rest of the demonstration as an exercise.

Ex. 842.—Inscribe an equilateral triangle in a given triangle ABC: have one side parallel to BC.

Solve as above.

Ex. 843.—Inscribe an equilateral triangle PQR in a given triangle ABC, having one side QR parallel to a given straight line.

Draw a parallel to the given straight line, meeting CA, AB *in* q, r. *On* qr *describe an equilateral triangle* pqr. *Join* Ap, *and let it cut* BC *in* P. *Draw* PQ, PR *parallel to* AB, AC.

Ex. 844.—Inscribe in a given triangle ABC a triangle similar to a given one, and having one side parallel to BC.

Ex. 845.—Inscribe in a given triangle ABC a triangle similar to a given one, and having one side parallel to a given straight line.

Ex. 846.—ABCDE is a regular pentagon, BEGF a square described on BE. If AF, AG cut BC, DE in f, g; show that fg is the side of a square begf whose corners are on sides ABCDE.

Ex. 847.—Draw a square whose corners shall be on sides of a given regular hexagon.

Ex. 848.—To find a point within a given isosceles triangle whose distance from each base angle is double its distance from the vertex.

Ex. 849.—In a given triangle inscribe a parallelogram whose sides are in a given ratio.

Ex. 850.—In a given triangle inscribe a rhombus having an angle equal to a given rectilineal angle.

The attention of the student is drawn to an important distinction between the *two possible kinds of similarity* which may exist between two given similar rectilineal figures. It may assist his comprehension to consider first the important special case of *congruence.*

If two congruent figures are (like the pair of triangles ABC, DEF in the diagrams of I. 4 or I. 8), such that superposition could be effected without taking either of them out of the plane in which they are, they are said to be **directly congruent.**

If two congruent figures are (like the pair of triangles AFC, AGB in the diagram of I. 5, or the pair ADF, AEF in the diagram of I. 9), such that superposition could be effected after one of them had been taken out of the plane in which they are, and turned over, they are said to be **inversely congruent.**

In the same way, if two similar figures are such that they can be 'similarly placed,' or 'oppositely placed,' without taking either of them out of the plane in which they are said to be **directly similar**; but if they are such that they could be similarly or oppositely placed after one of them had been taken out of the plane in which they are and turned over, they are said to be **inversely similar.**

PROPOSITION 4.

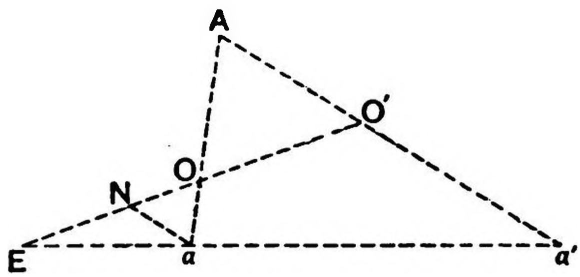
If two figures abcd, a'b'c'd' are each of them homothetic with a third figure ABCD, they are homothetic with each other, and the three centres of similitude are in a straight line.

Let O be the intl. centre, and m : M the ratio, of similitude of abcd, ABCD; O' and m' : M those of a b'c'd', ABCD.

Let OO', aa' meet in E, and on O'E take N such that ON : OO' :: m : M, and join Na,

$\because$ ON : OO' :: Oa : OA ;

$\therefore$ Na is $\parallel$ to Aa′,
and Na : AO′ :: m : M.



But AO′ : O′a′ :: M : m′;
$\therefore$ Na : O′a′ :: m : m′;
$\therefore$ EN : EO′ :: m : m′;
$\therefore$ aa′ cuts OO′ in a pt. E such that EN : EO′ :: m : m′.
Similarly bb′, cc′, dd′ cut OO′ at the same pt. E.
Also Ea : Ea′ :: EN : EO′
:: m : m′.
Similarly also Eb : Eb′, Ec : Ec′, Ed : Ed′ each = m : m′.
Hence E is a centre of similitude of abcd, a′b′c′d′.

A slight modification of the proof would be necessary if one or both of the two given centres of similitude were external.

DEF.—**The line passing through the three centres of similitude of three similar figures taken two at a time is called the 'axis of similitude' of the three figures.**

Ex. 851.—Use the diagram of Prop. 4 to show that if two similar rectilineal figures abcd, ABCD were respectively similarly and oppositely placed to a third similar rectilineal figure a′b′c′d′, they would be oppositely placed to one another.

Referring to the section on Loci (Prop. 6, p. 433), it is easy to see that—

(1) If a straight line drawn through the centre of similitude O of two homothetic rectilineal figures abcd, ABCD meet their boundaries at corresponding points p, P, the ratio Op : OP is the ratio of similitude of the two figures abcd, ABCD.

(2) If a straight line drawn through a fixed point O meet the boundary of a given rectilineal figure ABCD in P, and along OP (or PO produced) a point p be taken, such that Op : OP in a given ratio, the locus of p is the boundary of a rectilineal figure abcd homothetic with ABCD and having the given ratio for the ratio of similitude.

Thus the relationship between two homothetic rectilineal figures abcd, ABCD and their centre of similitude O is the same as that of two circles qpq′, QPQ′ (see p. 436), and the point O dividing the join of their centres externally or internally in the ratio of their radii.

Hence two circles are looked upon as homothetic figures, and the point O is called their centre of similitude.

Two such points can always be found for two given unequal circles which are not concentric.

For two equal circles the internal point of division only can be found, and it is a 'centre of symmetry' for the two circles.

For two concentric circles the common centre is the only centre of similitude.

If the two circles have a pair of external common tangents, their point of concurrence is the external centre of similitude.

If the two circles have also a pair of internal common tangents, their point of concurrence is the internal centre of similitude.

In general, two homothetic rectilineal figures do not, like two circles, have both an external and an internal centre of similitude.

If, however, they each have (like parallelograms) a centre of symmetry,' they will have two centres of similitude (an external and an internal), unless they are equal or concentric.

Ex. 852.—If two similar parallelograms have one centre of similitude, they must have two unless they are equal or concentric.

Ex. 853.—If two regular hexagons have one centre of similitude, they must have two unless they are equal or concentric.

What connection has Prop. 4 with this and the last exercise?

Ex. 854.—D, E, F are the mid points of the sides BC, CA, AB of a triangle ABC; D′, E′, F′ the mid points of the straight lines PA, PB, PC, joining its corners to a point P. Show that DD′, EE′, FF′ are concurrent.

Ex. 855.—A circle is described touching the side BC of triangle ABC, and the other two sides AB, AC produced. Show that it is the in-circle of a triangle AB′C′ similar to triangle ABC.

Ex. 856.—In the fig. of 'VI. B,' show that the circle described to pass through A and touch BC in D will touch the circle ABC at A.

Ex. 857.—Two similar and similarly placed triangles are such that the circum-circle of one is the in-circle of the other. Compare their areas. If one is four times the other, show that they are equilateral.

Ex. 858.—O is one of the points of intersection of two fixed circles; OPQ a straight line meeting the circles in P and Q; R divides PQ externally or internally in any given ratio. Find the locus of R.

If A *be other point of section, triangle* APQ *is of constant species.*

Ex. 859.—In two given circles external to each other, to inscribe two triangles similar to a given triangle, and having a side of one in the same straight line as the corresponding side of the other.

There are four solutions, two for each centre of similitude.

Ex. 860.—If two circles cut each other, to inscribe in the space between them a parallelogram of which one side is given.

PROPOSITION 5.

If from a given point O there be drawn straight lines Oa, Ob, Oc, Od, all bearing the same ratio (m : M) to the straight lines OA, OB, OC, OD, joining O to the corners of a given rectilineal figure ABCD, and all making with them the same angle θ in the same sense, then a, b, c, d are corners of a rectilineal figure abcd directly similar to ABCD.

Along OA, OB, OC, OD take Oa′, Ob′, Oc′, Od′ equal to Oa, Ob, Oc, Od.

Then by Prop. 1, a′b′c′d′ is similar to ABCD.

Now turn the figure formed by a′b′c′d′ and the st. lines, joining its corners to O round O until Oa′ falls along Oa.

Then Ob′, Oc′, Od′ will fall along Ob, Oc, Od, and a′, b′, c′, d′ will coincide with a, b, c, d.

We shall call the point O a '**centre of stretch-rotation**' for the two directly similar figures, abcd, ABCD.

The figure abcd may be said to be derived from the

figure ABCD by a 'stretch' m : M from O and a 'turn' θ round O.

In Prop. 1, abcd is derived from ABCD by a 'stretch' Oa : OA only.

In Prop. 2, abcd is derived from ABCD by a stretch Oa : OA and a straight angle 'turn' round O.

PROPOSITION 6.

A centre of stretch-rotation can be found for any two unequal directly similar rectilineal figures which are not homothetic.

For let ap, AP (diagm. on p. 435), corresponding sides of two directly similar figures, meet in C; and let the circum-⊙s of △s CaA, CpP cut again in O.

Then ∠ apO = int. ∠ APO of cyclic quadl. pCPO,
and ∠ Oap = ext. ∠ OAP of cyclic quadl. aCAO;
∴ △s Oap, OAP are equiangr.;
∴ Oa : Op :: OA : OP;
∴ Oa : OA :: Op : OP.

Let pq, PQ be the sides of the similar figures consecutive to ap, AP,
∵ ∠ apq = ∠ APQ, [Hyp.
and ∠ apO = ∠ APO;
∴ ∠ Opq = ∠ OPQ.

Again Op : pa :: OP : PA (∵ △s Oap, OAP are equiangr.);
and pa : pq :: PA : PQ; [Hyp.
∴ Op : pq :: OP : PQ.
But ∠ Opq = ∠ OPQ;
∴ Op : Oq :: OP : OQ.
Also ∵ ∠ aOp = ∠ AOP;
∴ ∠ aOA = ∠ pOP.
Similarly ∠ pOP = ∠ qOQ, and so on.

Hence O is a centre of stretch-rotation for the two similar figures.

The result of Prop. 4 in the section on Loci might be thus stated :—

The figure derived from a given circle by a stretch from a given point O and a given turn round O is a circle.

We shall speak of O as a centre of stretch-rotation of the two circles.

PROPOSITION 7.

Any pair of unequal circles which are not concentric have an indefinite number of centres of stretch-rotation which are all concyclic.

Let a, A be the centres of the two given ⊙s; O a pt. such that Oa : OA in the ratio of the radii ap, AP. (Fig. on p. 437.)

Take any pts. p, P on the ⊙s such that

∠ Oap, OAP are equal and in the same sense,

∵ Oa : OA :: ap : AP;

∴ Oa : ap :: OA : AP.

But ∠ Oap = ∠ OAP;

∴ Op : Oa :: OP : OA;

∴ Op : OP :: Oa : OA.

Also ∠ aOp = ∠ AOP;

∴ ∠ pOP = ∠ AOa;

∴ O is a centre of stretch-rotation for the two ⊙s.

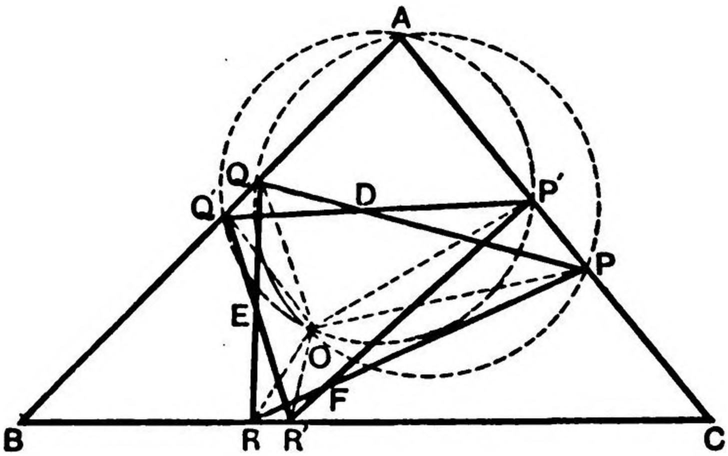
Similarly for other points whose distances from a, A are in the ratio of the radii.

But all such pts. lie on the ⊙ which has for a diamr. the join of the pts. dividing aA externally and internally in the ratio of the radii, *i.e.* on the join of the centres of similitude.

This ⊙ is called the '**circle of similitude**' of the two given ⊙s.

PROPOSITION 8.

PQR is a triangle inscribed in given triangle ABC. If the angles at P, Q, R on the sides CA, AB, BC are of given magnitude, the circum-circles of the triangles have a fixed point O in common.



Let P′Q′R′ be *any* second △ inscribed in △ABC, having ∠s P′, Q′, R′ equal to ∠s P, Q, R of △ PQR, and let the circum-⊙s of △s APQ, AP′Q′ intersect in O. Join OP, OQ, OR, OP′, OQ′, OR′. Let QR, Q′R′ cross at E.

Then ∠ OQQ′ = int. ∠ OPP′ of cyclic quadl. OPAQ,
and ∠ OQ′Q = ext. ∠ OP′P of cyclic quadl. OP′AQ′;

∴ △s OQQ′, OPP′ are equiangr.;
∴ OQ : OQ′ :: OP : OP′.
Again ∵ ∠ QOQ′ = ∠ POP′;
∴ ∠ QOP = ∠ Q′OP′,
and ∵ OQ : OQ′ :: OP : OP′;
∴ OQ : OP :: OQ′ : OP′;
∴ ∠ OQP = ∠ OQ′P′
and OQ : QP :: OQ′ : Q′P′ } [VI. 4.
But ∠ PQR = ∠ P′Q R′,
and PQ : QR :: P′Q′ : Q′R′;
∴ remg. ∠ OQR = remg. ∠ OQ′R′,

$$\text{and } OQ : QR :: OQ' : Q'R';$$
$$\therefore \angle OQR = \angle OQ'R',$$
$$\text{and } \angle ORQ = \angle OR'Q';$$

$\therefore$ the circum-⊙s of $\triangle$s QEQ', RER' pass through O;
$\therefore$ the circum-⊙ of $\triangle BQR$ passes through O (see p. 248);
$\therefore$ O is a fixed pt.

Again $\because$ $\angle ORC$ = int. $\angle OQB$ of cyclic quadl. $ORBQ$,
= int. $\angle OPA$ of cyclic quadl. $OPAQ$;
$\therefore$ circum-⊙ of $\triangle CPR$ passes through O.

Similarly the circum-⊙s of $\triangle$s $BQ'R'$, $CP'R'$ also pass through O

Note that since triangles OQR, $OQ'R'$ are similar,
$$OR : OQ :: OR' : OQ';$$
$$\therefore OR : OR' :: OQ : OQ',$$
$$:: OP : OP';$$
$$\text{also } \angle ROR' = \angle QOQ' = \angle POP';$$
$\therefore$ O is the centre of stretch-rotation for the two $\triangle$s PQR, $P'Q'R'$.

Hence:—**If any number of directly similar triangles be inscribed in a given triangle, so that all corresponding vertices lie on the same side, they have a common centre of stretch-rotation.**

Some interesting special cases of this general proposition may be noted.

(i) Suppose $\angle P = \angle A$; $\angle Q = \angle B$; $\angle R = \angle C$,

Then RP, PQ, QR are tangents to the circum-⊙s of $\triangle$s APQ, BQR, CRP;

$\therefore$ $\angle OPR = \angle OQP$ in alt. segt. $OPAQ$,
$= \angle ORQ$ in alt. segt. $OQBR$;

$\therefore$ the fixed point O is a 'Brocard point' for each of the directly similar $\triangle$s.

Again $\angle OAC = \angle OQP$ in same segt. $OQAP$,
$\angle OBA = \angle ORQ$,, $ORBQ$,
$\angle OCB = \angle OPR$,, $OPCR$;
$\therefore$ O is a 'Brocard point' of $\triangle ABC$.

This also follows from the fact that the $\triangle ABC$ itself belongs to the set of directly similar $\triangle$s.

(ii) If $\angle Q = \angle A$; $\angle R = \angle B$; $\angle P = \angle C$,
O is a Brocard point for each of the $\triangle$s,
such that $\angle OAB = \angle OBC = \angle OCA$,
$= \angle OPQ = \angle OQR = \angle ORP$.

(iii) If $\angle R = \angle A$; $\angle P = \angle B$; $\angle Q = \angle C$,

O is the circum-centre of ABC and the ortho-centre of $\triangle$ PQR;

$\therefore$ the common chd. PQ of the circum-$\odot$s of $\triangle$s PAQ, PRQ subtends equal $\angle$s PAQ, PRQ;

$\therefore$ circum-$\odot$ of $\triangle$ APQ = circum-$\odot$ of $\triangle$ PQR.

So also circum-$\odot$s of $\triangle$s BQR, CRP each = circum-$\odot$ of $\triangle$ PQR;

$\therefore$ chds. OA, OB, OC subtending equal $\angle$s OPA, OQB, ORC in equal $\odot$s are equal.

$$\begin{aligned} \text{Again } \triangle OBA &= \angle OAB, \\ &= \angle OPQ, \\ \text{and } \angle AOB &= 2\angle C, \\ &= 2\angle PQR; \\ \therefore 2\angle OPQ + 2\angle PQR &= \angle\text{s } OBA, OAB, AOB, \\ &= 2 \text{ rt. } \angle\text{s}; \\ \therefore \angle OPQ + \angle PQR &= \text{a rt. } \angle; \end{aligned}$$

$\therefore$ PO is $\perp$r to QR.

Similarly QO, RO are $\perp$r to RP, PQ.

(iv) The case of a straight line cut by the sides CA, AB, BC in P, Q, R, so that the segments PQ, QR, RP have given ratios to one another is a 'limiting case' of the general theorem.

We leave the special investigation of this case as an exercise.

The previous theorems on sets of directly similar triangles with a common centre of stretch-rotation have a close connection with the geometry of the parabola.

Some further theorems concerning such sets of triangles will be given in the next section (on **Maxima and Minima**).

Ex. 861.—PQR is a triangle inscribed in a given triangle ABC. Show how to inscribe in triangle ABC any number of triangles directly similar to triangle PQR.

Ex. 862.—O is a Brocard point of triangle ABC; OP, OQ, OR are drawn to CA, AB, BC, so that angle OPA = angle OQB = angle ORC. Show that triangle PQR is directly similar to triangle ABC.

Show also that O is a Brocard point of triangle PQR.

Ex. 863.—O is the circum-centre of triangle ABC; OP, OQ, OR are drawn to BC, CA, AB, so that angle OPA = angle OQB = angle ORC. Show that triangle PQR is directly similar to triangle ABC.

Show also that O is the ortho-centre of triangle PQR.

Ex. 864.—A straight line cuts the sides CA, AB, BC in P, Q, R. Show how to draw any number of straight lines which shall be cut into segments having the same ratio as PQ, QR, RP.

From O *the common point of intersection of the circum-circles of triangles* APQ, BQR, CRP (*see p.* 248) *draw* OP′, OQ′, OR′ *to* CA, AB, BC, *so that angles* POP′, QOQ′, ROR′ *are equal and in the same sense;* P′, Q′, R′ *will lie in a straight line such as is required.*

Ex. 865.—In fig. of Prop. 8,

Perimr. of $\triangle$ PQR : perimr. of $\triangle$ P′Q′R′ :: OP : OP′.

Ex. 866.—In fig. of Prop. 8,

$\triangle$ PQR : $\triangle$ P′Q′R′ :: sq. on OP : sq. on OP′.

Ex. 867.—In a given triangle ABC, to inscribe a triangle PQR whose species and perimeter are given.

Ex. 868.—In a given triangle ABC to inscribe a triangle PQR whose species and magnitude are given.

Principia, Book I., Lemma xxvi.

Ex. 869.—'A right line may be drawn, whose parts, given in length, may be intercepted between three right lines, given in position.'

Principia, Book I. Lemma xxvi., Cor.

Ex. 870.—All the corresponding points of a set of directly similar triangles inscribed in a given triangle, as in Prop. 8, lie in a straight line except the points corresponding to O, which coincide with O.

Ex. 871.—A straight line cuts the sides CA, AB, BC of a given triangle ABC in P, Q, R, such that the ratios of PQ, QR, RP are given. A point divides S, one of these segments, into parts having a given ratio to one another. Show that the locus of S is a straight line.

Ex. 872.—'To describe a trapezium whose species is given and whose several angles may respectively touch four right lines given in position, which are neither all parallel nor converge to a common point.'

Principia, Book I. Lemma xxvii.

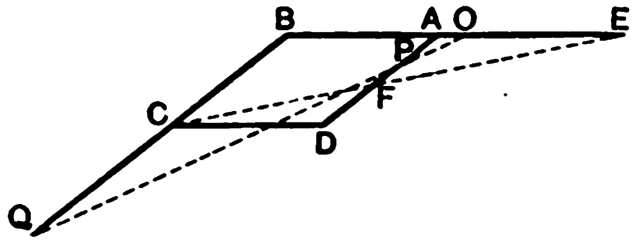
Ex. 873.—'A right line may be drawn whose parts, intercepted in a given order between four right lines given in position, shall have a given proportion to each other.'

Principia, Book I., Lemma xxvii., Cor.

Ex. 874.—D is any point in the base BC of a triangle ABC: from D are drawn DE, DF parallel to two given straight lines to meet AB, AC

in E, F. Show that the circum-circle of triangle ADE has a fixed chord, and hence that the locus of its centre is a straight line. *See Exs.* 664, 665.

The 'Pantagraph' is an instrument whose use depends on the properties of similar figures.



It consists of four rods EAB, BCQ, CD, DA pivoted together at A, B, C, D, and such that CD=AB and DA=BC. Hence for all possible movements of the instrument, ABCD is a ||gm.

Suppose O, P, Q to be fixed points on EB, AD, BQ, such that in some position of the instrument O, P, Q are in a st. line.

Then OA : AP :: OB : BQ.

But ∠OAP=∠OBQ for all positions.

∴ ∠AOP=∠BOQ;

∴ O, P, Q are always in a st. line, and OP : OQ=OA : OB always.

If therefore O remains fixed and Q describes any figure, P will describe a similar and similarly placed figure reduced in the ratio OA : OB.

It is obvious that the instrument could also be used for enlarging.

Hence O is a 'centre of similitude' of the original figure and its enlarged or reduced copy.

For a modification of the above instrument, by Professor Sylvester, which can 'turn' as well as 'stretch' a given figure (*i.e.* in which O is a centre of 'stretch-rotation'), see *How to Draw a Straight Line*, p. 23, '*Skew Pantagraph.*'

The principle of the common pantagraph is used in the parallelogram frequently attached to the oscillating beam of a steam-engine.

Let E be the centre of the beam, ABCD the jointed parallelogram, and let the corner D be joined to a fixed centre H by a rod not shown in the figure, so that D is forced to move on a circular arc convex towards A. Then as A describes an arc about E convex towards D, it occurred to Watt that a point F might be found which, for moderate oscillations, moved approximately in a straight line. This is the so-called 'parallel motion' in which no use is necessarily made of the parallelogram ABCD.

But if the parallelogram ABCD be so constructed that EFC are in a straight line for one position, they remain so for all positions, and by the principle of the pantagraph C also describes an approximately straight path. In Watt's engine C was attached to the end of the piston-rod which drives the engine, F to that of the exhaust-pump.

MAXIMA and MINIMA.

DEF.—**When a geometrical magnitude X, which varies continuously according to any given law, passes through a certain value M greater than the values immediately preceding or immediately succeeding it, the magnitude X is said to be a 'maximum' when it has that value M.**

Thus suppose a pt. D to move along the st. line AB from the end A through the mid pt. C to the end B; the value of

the rectangle AD.DB varies continuously with the position of D: also since

rect. $AD.DB = CA^2 - CD^2$, [II. 5.

it is clear that when D is at C that value is greater than when D is moving towards C or away from C. Hence the rectangle AD.DB is said to be a *maximum* when D coincides with C.

DEF.—**When a geometrical magnitude X which varies continuously according to any given law passes through a certain value M less than the values immediately preceding or immediately succeeding it, the magnitude X is said to be a 'minimum' when it has that value M.**

Thus suppose a pt. D to move along the st. line AB from the end A through the mid pt. C to the end B; the value of $AD^2 + DB^2$ varies continuously with the position of D; also since

$$AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2,$$

it is clear that when D is at C that value is less than when D

is moving towards C or away from C. Hence AD^2+DB^2 is said to be a *minimum* when D coincides with C.

It does not follow that when a magnitude is a maximum or minimum, according to the above definitions, that it has assumed the absolutely greatest or the absolutely least value it is capable of assuming: for it may increase and then decrease; then increase again, and so on. Whenever it has left off increasing, and is just about to decrease, it is by definition a maximum; and whenever it has left off decreasing and is just about to increase, it is by definition a minimum.

Thus a varying magnitude may have several maximum and several minimum values.

Note also that **maxima and minima occur alternately.**

In our first illustration, if we confine ourselves to the finite segment AB of the indefinite straight line through A and B, the rectangle AD.DB is an absolute maximum when D is at C; but if we suppose D to move from some point on BA produced, through A, C, B, and then along AB produced, it can be seen that rect. AD.DB decreases as D approaches A; that it increases as D moves from A to C; decreases as D moves from C to B; and increases again as D moves along AB produced: also, since rect. $AD.DB=CD^2-CB^2$, when D is in either produced part, its value may be made as great as we please by taking D further and further from C; hence the rect. AD.DB is not an *absolute* maximum when D is at C: it is a *relative* maximum occurring between two minima.

Note that these minima (D at A or B) give *zero* values.

In our second illustration it is plain that, whether we consider the finite segment AB or the indefinite st. line through A and B, AD^2+DB^2 is an absolute minimum when D is at C; since for all positions

$$AD^2+BD=2\,AC^2+2\,CD^2. \qquad \text{[II. 9, 10.}$$

We proceed to enunciate some propositions on maxima and minima whose truth has been already demonstrated in, or follows easily from, some of Euclid's propositions.

PROPOSITION 1.

The straight line drawn from a given point to a given straight line is a minimum when it is perpendicular to it. (I. 16, 19.)

This is demonstrated in III. 16.

PROPOSITION 2.

The sum of the straight lines drawn from a point to two given points is a minimum when that point lies in their join. (I. 20.)

PROPOSITION 3.

If a finite straight line be divided into two segments, the rectangle contained by them is a maximum when they are equal.

Or thus:—

If the sum of two straight lines be given, the rectangle contained by them is a maximum when they are equal.

Cor.—**Of all rectangles with the same perimeter, the square has the greatest area.**

PROPOSITION 4.

If a finite straight line be divided into two segments, the sum of the squares on them is a minimum when they are equal. (II. 9.)

An easy extension of this may be stated thus:—

The sum of the squares of the distances of a point from two given points is a minimum when it bisects their join. (II. 9, Ex. 203.)

PROPOSITION 5.

If the rectangle contained by two straight lines is given in magnitude, the sum of the squares on them is a minimum when they are equal, *i.e.* when the rectangle is a square.

For if $AB = BC$,
$AB^2 + BC^2 = 2\,AB.BC$.
If not, apply BC to AB,
then $AB^2 + BC^2 > 2\,AB.BC$. [II. 7.

PROPOSITION 6.

If the rectangle contained by two straight lines is given in magnitude, their sum is a minimum when they are equal, *i.e.* when the rectangle is a square.

For if $AB = BC$,
$(AB + BC)^2 = 4\,AB.BC$.
If not, apply BC to AB,
then $(AB + BC)^2 > 4\,AB.BC$. [II. 8.

COR.—**Of all rectangles having the same area, the square has the least perimeter.**

PROPOSITION 7.

If a straight line be drawn to the circumference of a circle from a given point within it which is not the centre, it is a maximum when it passes through the centre and a minimum when it forms the remaining part of the diameter through the given point. (III. 7.)

PROPOSITION 8.

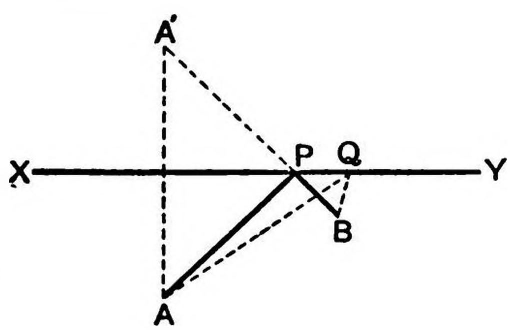
If a straight line be drawn to the circumference of a circle from a given external point, it is a maximum when it passes through the centre and a minimum when it forms that part of the maximum which is outside the circle. (III. 8.)

PROPOSITION 9.

The diameter is the maximum chord of a circle.

PROPOSITION 10.

If from two given points A and B on the same side of a given indefinite straight line XY, straight lines AQ, BQ are drawn to a point PQ in XY, their sum is a minimum when they make equal angles with XY.



Draw AO ⊥r to XY, and produce it to A′, so that OA′=OA. Join A′B, cutting XY in P. Take any pt. Q in XY. Join AP, AQ, A′Q.

In △s AOP, A′OP $\begin{cases} \text{AO, OP} = \text{A'O, OP,} \\ \text{and } \angle\text{AOP} = \angle\text{A'OP.} \end{cases}$

∴ AP=A′ .

Similarly AQ=A′Q

$$\begin{aligned}
\text{Also } \angle APX &= \angle A'PO, \\
&= \angle BPY. \\
\because AQ &= A'Q; \\
\therefore AQ+QB &= A'Q+QB. \\
\text{Similarly } AP+PB &= A'P+PB, \\
&= A'B.
\end{aligned}$$

But if Q does not coincide with P,

$$A'Q+QB > A'B; \qquad [I.\ 2.$$

$$\therefore AQ+QB > AP+PB,$$

The solution thus depends on Prop. 1.

We have demonstrated that $AQ+QB$ is an absolute minimum when Q coincides with P; and it is worthy of notice that as Q recedes from P in either direction along XY, $A'Q+QB$ continually increases, so that *there is no other relative minimum or maximum*; and that if two points on XY give equal values for $AQ+QB$, they must be *on opposite sides of* P.

This proposition has interesting applications in Optics.

PROPOSITION 11.

Of all equivalent triangles on the same base, the isosceles has the minimum perimeter.

The vertices of any No. of equivt. $\triangle$*s on the same base* AB, *and on the same side of it, lie on an indefinite st. line* XY || *to* AB; *and it can be shown by a construction and demonstration like that of Prop.* 10 *that the sum of the sides is a minimum when they make equal* $\angle$*s with* XY, i.e. *when the* $\triangle$ *is isosceles*

PROPOSITION 12.

A and B are two fixed points; Q a point in a given indefinite straight line XY. To find when AQ^2+QB^2 is a minimum.

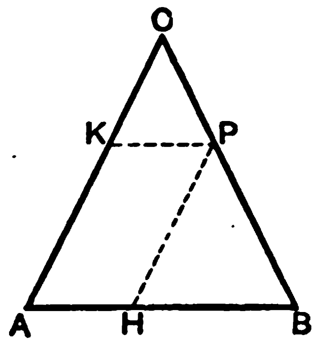
Bisect AB in O. Then $AQ^2+QB^2=2\,AO^2+2\,OQ^2$;

∴ AQ^2+QB^2 is a minimum when OQ is a minimum, *i.e.* when OQ is ⊥ to XY.

Compare Ex. 278.

PROPOSITION 13.

From a point P in the side BC of a triangle ABC, PH, PK are drawn parallel to CA, AB so as to form a parallelogram AHPK. It is required to find when AHPK is a maximum.



∵ △s BHP, BAC are equiangr.;

∴ PH : PB :: AC : CB.

Similarly PK : PC :: AB : CB;

∴ PH.PK : PB.PC :: CA.AB : CB^2;

∴ rect. PH.PK is a maximum when rect. PB.PC is a maximum, *i.e.* when P bisects BC.

∴ ||gm AHPK is a maximum when P bisects BC, *i.e.* when it is half of the triangle ABC.

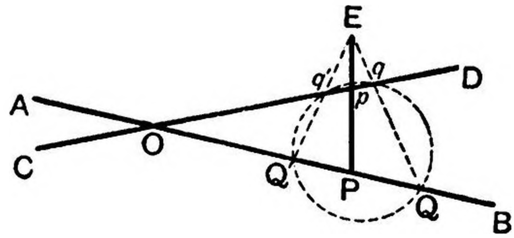
Note that the maximum area of ||gm AHPK is half the area of ABC, and that △ABC is never less than double of ||gm AHPK.

Many problems in Maxima and Minima may be solved like the problems in Props. 10-13, by reducing to one of the 'Standard Cases' given in Props. 1-9.

We append some of these as exercises for the student.

If, however, the student fails in reducing a given problem to one of the 'Standard cases,' he should have recourse to the method of '*coincidence of equal values*' which we proceed to exemplify in the solution of two problems. A formal statement of the principles on which the method depends will be afterwards made.

AOB, COD are two given intersecting straight lines; E, a given point. Through E is drawn a straight line cutting AB, CD in Q, q. It is required to find when the rectangle Eq.EQ is a minimum.



Here Eq.EQ may be made as large as we please by taking EQ nearly || to AB or CD. Hence as the value of the rectangle changes gradually as we turn the line EQ round E, there must be two positions of minimum value.

Let EpP be one of them.

Then there must exist positions EqQ, Eq'Q' *on opposite sides of* EpP, *such that rect.* Eq.EQ=*rect.* Eq'.EQ';

∴ Q, q, q', Q' are concyclic;

∴ ∠OQ'q' = ∠OqQ.

Now let EqQ *move up to and coincide with* EpP; *then* Eq'Q' *simultaneously moves up to and coincides with* EpP;

∴ in the limit ∠OPp = ∠OpP,

i.e. EpP makes equal angles with AB and CD.

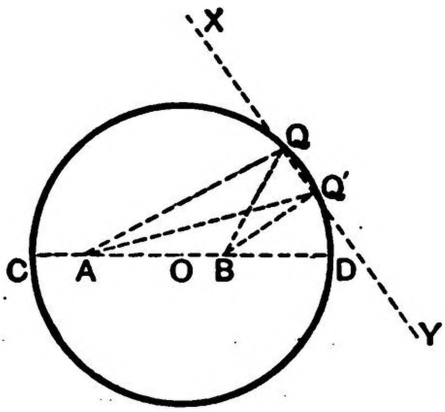
The other position of minimum is given by the other st. line through E, making equal angles with AB, CD.

The student should examine carefully the validity of the italicised portion of the above investigation.

PROPOSITION 14.

A and B are two points on the diameter COD of a circle CQD on opposite sides of the centre O, such that AO is greater than OB. To find when AQ + QB is a maximum or a minimum.

It is evident that when Q is at either end of the diamr. COD, AQ+QB is a maximum or a minimum. For if, as Q



approaches D on one side, AQ+QB increases, it must, from the symmetry of the figure, decrease as Q moves away from D on the other, and *vice versâ*. But this argument leaves unsettled whether D gives a maximum or a minimum, and the same may be said of C.

In what follows, when we know that a value is either a maximum or a minimum value, but have not demonstrated which, we shall speak of it as a '**turning value.**' There may be other positions of Q besides C and D which give *turning values* of AQ+QB, as to which the above simple considerations of symmetry give no information. These we proceed to investigate.

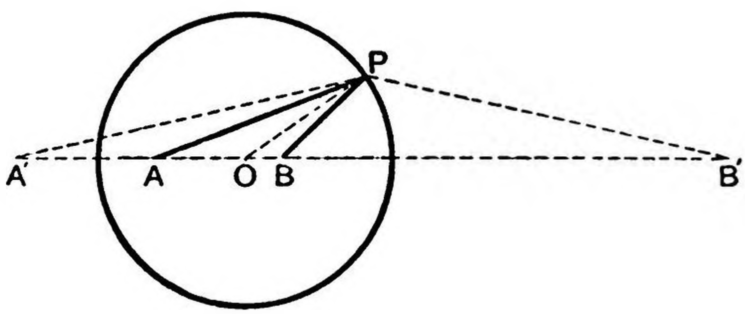
If there does exist a point on the ⊙ CQD for which AQ+QB has a 'turning value,' *there must exist two other*

points Q *and* Q′ *on the* ⊙, *on opposite sides of that pt., such that* AQ+QB=AQ′+Q′B.

Suppose this to be the case. Join Q, Q′ and produce to X and Y.

Then, by Prop. 10, there must exist on the st. line XY a pt. P *between* Q *and* Q′, such that ∠ APX= ∠ BPY.

Now let Q and Q′ move up and coincide with the supposed pt. of 'turning value' on the ⊙.



Then P will also coincide with it, and XY will be the tangent at P.

Hence at a pt. P of 'turning value' on the ⊙, the tangent at P and also the radius OP is equally inclined to AP and BP.

This condition is obviously satisfied at the points C and D already known to be points of turning value.

If it is satisfied at any other point P,

$$AP : PB :: AO : OB.$$

Now as Q moves round the ⊙ from C to D, the ratio AQ : QB is continually increasing from AC : CB, which is of less inequality to AD : DB, which is of greater inequality.

Hence if AO : OB > AD : DB, there is no point P such that AP : PB :: AO : OB.

In this case C and D are the only pts. which give turning values. We leave it to the student to show that C gives a minimum and D a maximum.

But if AO : OB < AD : DB, there will be such a point.

We shall leave it to the student to show that it gives a maximum value for AQ+QB, and that, *since maxima and minima occur alternately*, C and D must each give a minimum.

It can be found by Ex. 697.

The student's attention is drawn to the following interesting property on account of its connection with the more general problem known as **Alhazen's**, in which the positions of A, B are unrestricted.

Produce OA, OB to A′, B′, such that

$$OA.OA' = OB.OB' = OP^2.$$

Then OP touches circum-⊙ of △APA′;

$$\therefore \angle OPA = \angle OA'P.$$

Similarly $\angle OPB = \angle OB'P$;

$$\therefore \angle OA'P = \angle OB'P;$$

$$\therefore A'P = B'P;$$

∴ P lies on ⊥r bisector of A′B′.

The principle we have used in solving Props. 13 and 14 may be formally enunciated thus:—

On opposite sides of a position which gives a 'turning value' to a continuously varying magnitude, and indefinitely near to it, there must always exist two positions which give it equal values.

Conversely:—

Between every two positions which give equal values to a continuously varying magnitude, a position of turning value must always exist.

On the power which the application of this principle gives in solving problems on maxima and minima, we may quote the following passage from Tait and Steele's *Dynamics of a Particle*:—

'*This principle, though excessively simple, is of very great power, and often enables us to solve problems of Maxima and Minima such as require in Analysis not merely the processes of the Differential Calculus, but those of the Calculus of Variations.*'

Ex. 875.—AOB is a given diamr. of a ⊙ AQB, whose centre is O;

M any pt. on AB; MQ ⊥r to AB. Find when rect. OM.MQ is a maximum.

Use Prop. 13 *to show that, if the tangent at the point* P *which gives a maximum meets* OB *produced at* T *and the perpendicular to* AB *through* O *in* R, RP=PT.

Ex. 876.—Inscribe the greatest square in a given quadrant AOB of a circle with its sides along OA, OB. *Use Prop.* 13.

Ex. 877.—Inscribe the greatest rectangle in a semi-circle AQB having one side on AB. *Use Prop.* 13.

Ex. 878.—AB is a given diamr. of a ⊙ AQB; M any pt. on AB; MQ ⊥r to AB. Find when rect. AM.MQ is a maximum. See hint for Ex.

Ex. 879.—A is a given point on a given ⊙ AQB; AY ⊥r to the tangent at Q. Find when Y is furthest from the diamr. through A.

Draw QM, YN ⊥r to diamr. Show that YN is propl. to △AMQ. Apply last exercise.

Ex. 880.—O is a given pt. on a given st. line OX; Q any pt. on another given st. line, cutting OX at T; QM ⊥r to OX. Show that when $OM^2 \backsim MQ^2$ is a maximum, MQ is a tangent to the circum-⊙ of OTQ, and give a geometrical construction for finding Q.

Taking two pts. Q, Q′, such that $OM^2 - MQ^2 = OM'^2 - M'Q'^2$, and drawing QR ⊥r to M′Q′; show that

$$OM + OM' : MQ + M'Q' :: RQ' : MM',$$

and hence in the limit

$$OM : MQ :: MQ : TM.$$

Ex. 881.—C is a given pt. on a ⊙; Q any pt. on either of the quadrantal arcs terminating at C; QM ⊥r to the tangent at C. Show that the maximum value of $CM^2 - MQ^2$ is half the square of the radius.

Ex. 882.—A and B are two fixed pts. on two given ⊙s APC, BQD; AP, BQ are ∥ chords. Show that when rect. AP.BQ is a maximum, AP 'corresponds' to BQ.

Ex. 883.—OX, OY are two given st. lines; A a given pt. within ∠XOY. Show that the line through A, which forms with OX, OY the △ of minimum area, is bisected at A.

Use Ex. 684, *or apply the result of Prop.* 13.

In applying the method of *Coincidence of Equal Values*, a

knowledge of Loci is often useful; as in the solution of the following problems:—

PROPOSITION 15.

A and B are two given points; Q a point on a given circle. Find when $AQ^2 \backsim QB^2$ is a maximum or minimum.

If a st. line ⊥r to AB meet the ⊙ in Q, Q′,

$$AQ^2-QB^2=AQ'^2-Q'B^2;$$ [Ex. 194.

and ∴ a turning value must exist on each side of this ⊥r.

Making Q and Q′ coincide, we see that the points which give a turning value to $AQ^2 \backsim QB^2$ are the points of contact of tangents to the ⊙ which are ⊥r to AB.

If the ⊥r bisector of AB cuts the ⊙ in CD, the 'turning values' will be both *maxima*; C and D giving *zero minima*.

If the ⊥r bisector of AB does not meet the ⊙, one of these turning values will be a *maximum* and the other a *minimum*.

PROPOSITION 16.

A and B are two given points; Q a point on a given circle. When is triangle AQB a maximum or a minimum?

If a ∥ to AB meets the ⊙ in Q, Q′,

$$\triangle AQB=\triangle AQ'B.$$

Hence there must be a 'turning value' on each side of this ∥; and, making these equal values coincide, we see that the points which give these 'turning values' are the pts. of contact of tangents to the ⊙ which are ∥ to AB.

If AB cuts the ⊙ in C, D, these values are both *maxima*; C and D giving *zero minima*.

If AB does not cut the ⊙, one of them is a *maximum* and the other a *minimum*.

We add some general directions for the use of the method

employed in Props. 15 and 16 in solving the general problem :—

To find the position of a point Q on a given straight line, or circle X, at which a certain varying magnitude M, depending upon the position of Q, is a maximum or a minimum.

First disregard the st. line X *and consider the locus* L *of the pt.* Q, *when the magnitude* M *is kept constant.*

Then let this constant be so chosen that L *touches* X.

The point of contact P *will be a position of turning value.*

Exs. 492, 494 may be solved immediately by this method, for if ∠ AQB remains constant, the locus of Q is a ⊙ through A, B : hence describe a ⊙ through the two given pts. A, B to touch the given st. line or circle. The pt. of contact will make ∠ AQB a maximum or minimum.

Similarly a knowledge of Envelopes may be applied to the solution of the general problem :—

To find the position of the straight line Q, which passes through a given point X, in which a certain varying magnitude M depending upon the position of Q is a maximum or a minimum.

First disregard the point X *and consider the envelope* E *of the st. line* Q, *when the magnitude* M *is kept constant.*

Then let this constant be so chosen that E *passes through* X.

The tangent at X *to the envelope will be in a position of turning value.*

Ex. 884.—A and B are two given points. Find a pt. Q such that AQ : QB shall be a given ratio, and Q at its greatest possible distance from AB.

Ex. 885.—AB, AC are two given finite st. lines. Show that the locus of a pt. Q which makes △ QAB + △ QAC constant is a straight line ∥ to BC. Hence show how to find a pt. Q on a given ⊙ such that △ QAB + △ QAC shall be a maximum or a minimum.

Ex. 886.—APX, AQY are two given st. lines. If the perimeter of △ APQ is kept constant, show that PQ envelopes a certain ⊙.

Hence find a st. line PQ, which passes through a given pt. C and cuts off a $\triangle$ APQ from AX, AY of minimum perimeter.

Show also how to find a st. line PQ which shall touch a given $\odot$ and cut off a $\triangle$ of minimum perimeter from AX, AY.

Many problems on Maxima and Minima lie beyond the range of Elementary Geometry. For, though the method of *Coincidence of Equal Values* is applicable to all cases, and will generally teach us some property of the figure on whose construction the solution depends, it may happen that the actual construction of the figure cannot be effected by 'rule and compass' only.

Thus suppose the two given points A, B in Prop. 14 were not on the same diamr. of the given $\odot$.

We could show as before that, if P is the pt. on the given $\odot$ at which Q gives a 'turning value' to $AQ+QB$, that OP is equally inclined to AP, BP, and subtends equal angles at the 'inverse points' A' and B' of A and B (see p. 251).

But this will not in general enable us to find P. We see that P must lie on a locus such that

$$\angle PA'B' - \angle PB'A' = \angle OA'B' - OB'A',$$

but this locus is a straight line only when $\angle OA'B' = \angle OB'A'$. In other cases it is a curve whose construction cannot be effected by 'rule and compass.' (The locus is a rectangular hyperbola, Newton's *Arithmetica Universalis*, Prob. xli.)

PROPOSITION 17.

To divide a given straight line AB into three parts such that the sum of the squares on them may be a minimum.

If $AP^2+PQ^2+QB^2$ is the least possible,

$$AP=PQ=QB.$$

For if AP is not equal to PQ, *keeping* QB *fixed*, we might move P so as to decrease AP^2+PQ^2.

Similarly $PQ=QB$.

The method can be extended to any number of segments.

PROPOSITION 18.

To find a point within a triangle ABC such that the sum of its distances from A, B, C is a minimum.

If AQ+BQ+CQ *is a minimum*,
∠BQC=∠CQA=∠AQB.

For *keeping* CQ *of constant length* Q describes a ⊙, and if AQ+BQ is a minimum, it may be shown, as in Prop. 13, that ∠BQC=∠CQA. Similarly ∠CQA=∠AQB.

It may happen that no point within the △ gives a turning value.

PROPOSITION 19.

In a given triangle ABC to inscribe a triangle whose perimeter is a minimum.

If PQ+QR+RP *is a minimum*, AB, BC, CA must be the extl. bisector of ∠s of △PQR.

Keep one side fixed and use Prop. 10.

It may happen that there is no inscribed △ of minimum perimeter.

The method can be extended to polygons.

Ex. 887.—Find a pt. Q within a △ABC such that $AQ^2+BQ^2+CQ^2$ is a minimum.

Ex. 888.—Q is a pt. within a △ABC; QD, QE, QF are drawn ⊥r to BC, CA, AF. If $QD^2+QE^2+QF^2$ is a minimum, show that D is the centroid of △DEF, and hence that it is the symmedian pt. of △ABC.

The remaining propositions deal chiefly with the properties of *Isoperimetrical Figures*, *i.e.* of figures having equal perimeters.

PROPOSITION 20.

Of all triangles having the same perimeter, the equilateral has the greatest area.

Keeping one side fixed, we see that the other two must be equal.

Cor.—**Of all isoperimetrical figures with a given number of sides, the equilateral is the greatest.**

PROPOSITION 21.

If two sides of a triangle are given in magnitude, its area is a maximum when those two sides contain a right angle.

The proof is left to the student.

Cor.—**Of isoperimetrical parallelograms the square has the greatest area.**

For the maximum parallelogram must be equilateral by Prop. 20 *and rectangular by Prop.* 21.

PROPOSITION 22.

If ABCD is the greatest quadrilateral that can be described with the sides AB, BC, CD of given lengths, A, B, C, D must lie on a circle, of which the fourth side AD is the diameter.

If $\angle$ ABD were not a rt. $\angle$, keeping $\triangle$BCD fixed, we could make $\triangle$ABD greater by putting AB $\perp$r to BD (Prop. 21.)

Hence $\angle$ ABD, and similarly $\angle$ ACD, must be right.

By similar reasoning it can be shown that—

If ABC . . . HK is the greatest polygon that can be described with the sides AB, BO, CD . . . HK of given lengths, A, B, O, . . . H, K must lie on a circle, of which AK is the diameter.

It may save the student some trouble to state that the *construction* of a cyclic quadrilateral ABCD whose three sides AB, BC, CD shall be respectively equal to three given straight lines, and whose fourth side AD shall be the diameter of the circum-circle, is beyond the range of Elementary Geometry, unless two of the given st. lines are equal.

It is, however, easy to prove that only one form of such quadrilaterals can exist.

For if in a $\odot$ ABCD, whose diamr. is AD, we draw any three chds. AB, BC, CD, and place in a greater or smaller circle three chds. ab, bc, cd, equal respectively to AB, BC, CD, ad cannot be a diamr. of that $\odot$.

Now of all the quadrilaterals with three sides, ab, bc, cd respectively equal to AB, BC, CD, there must be one whose form makes it contain the greatest possible area, and it cannot be any other than ABCD.

PROPOSITION 23.

A cyclic quadrilateral ABCD is greater than any quadrilateral abcd which is not cyclic, whose sides are equal to the corresponding sides of ABCD.

For let the other end E of the diamr. through A be on the arc CD. Join CE, ED, and on cd construct a $\triangle$ ced directly congruent with $\triangle$ CED, and join ae.

Then $\because$ abcd is not cyclic;

$\therefore$ $\angle$s abc, ace, adc cannot all be rt. $\angle$s.

Therefore one of the two figures abce, ade must be less than the corresponding part of ABCED, while the other cannot be greater;

$\therefore$ abced $<$ ABCED.

But $\triangle$ced $=$ $\triangle$CED;

$\therefore$ abcd $<$ ABCD.

N.B.—It is possible to construct by Elementary Geometry a cyclic quadrilateral whose sides shall be equal to those of a given quadrilateral.

PROPOSITION 24.

If a figure is the greatest that can be contained within a perimeter of given length, it must be a circle.

Inscribe any quadl. ABCD in it.

If A, B, C, D were not concyclic, we could, without altering the external parts cut off by its sides, make the quadl. ABCD, and therefore the whole figure, greater.

Similarly any other four points on the perimeter are concyclic.

Ex. 889.—A figure ABC is to be contained by a straight line AC whose length is not assigned while the remaining part ABC of the boundary is to be of given length. Show that when it is a maximum it must be a semicircle.

Ex. 890.—If a figure is the greatest that can be contained by a given straight line and another boundary of given length, it must be a segment of a circle.

Ex. 891.—A figure ABCDEF is to be contained by straight lines AB, CD, EF of given lengths. The *lengths* only of the intermediate parts BC, DE, FA are given. If the figure has its area a maximum, show that AB, CD, EF must be chords of a circle of which BC, DE, FA are arcs.

COMPOUND RATIO.

Referring to the definition of compound ratio in Book V., it is easy to see that if we first compound A : B and C : D, and then compound E : F with the resulting ratio, we get the same ratio as if we first compounded C : D and E : F, and then compounded A : B with the resulting ratio. Hence :—

PROPOSITION 1.

The ratio compounded of any given set of ratios is independent of the order in which they are taken.

In what follows the magnitudes considered are supposed to be straight lines.

By VI. 23 (see p. 397), the ratio compounded of the ratios

$$\left\{\begin{matrix} a : x \\ b : y \end{matrix}\right\}$$

$$= \text{rect. a.b} : \text{rect. x.y}$$

$$= \text{ratio compounded of } \left\{\begin{matrix} a : y \\ b : x \end{matrix}\right\};$$

$$\therefore \text{ ratio compounded of } \left\{\begin{matrix} a : x \\ b : y \\ c : z \end{matrix}\right\} = \text{ratio compd. of } \left\{\begin{matrix} a : x \\ b : z \\ c : y \end{matrix}\right\}$$

$$= \text{ratio compd. of } \left\{\begin{matrix} a : z \\ b : x \\ c : y \end{matrix}\right\}$$

and so on. Hence :—

PROPOSITION 2.

If from any such set of given ratios we form a new set by making any interchanges of consequents (or any interchanges of antecedents) the ratio compounded of the set is unaltered.

For the sake of brevity, the ratio compounded of $\left\{\begin{matrix} a : x \\ b : y \\ c : z \end{matrix}\right\}$

is frequently denoted thus, a.b.c : x.y.z. This notation can be used for any number of ratios.

When the ratio obtained by compounding the set is one of equality, the result is expressed thus, a.b.c=x.y.z.

The student is requested to observe that *no geometrical meaning is here assigned to such a symbol as* a.b.c or a.b.c.d.

PROPOSITION 3.

If a.b.c.d=p.q.r.s and b=r, then a.c.d=p.q.s.

For a.c.d.b=p.q.s.r.

PROPOSITION 4.

If a.b.c.d.e=p.q.r.s.t and a.b.d.=q.s.t, then c.e=p.r.

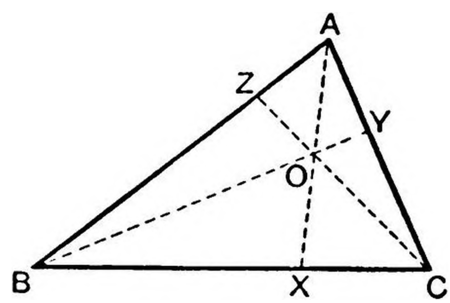
For c.e.a.b.d=p.r.q.s.t.

The student will probably notice that these propositions lead to the same rules for deducing equalities of compound ratios as would apply *if* a.b.c.d, *etc., were algebraical products.*

MISCELLANEOUS PROPOSITIONS.

PROP. 1.—If the straight lines AO, BO, CO joining the vertices of a triangle ABC to a point O be produced to cut BC, CA, AB in X, Y, Z, then the ratio compounded of the ratios AZ : ZB, BX : XC, CY : YA is one of equality, *i.e.* AZ.BX.CY = ZB.XC.YA. (Ceva's Theorem).

For AZ : ZB :: $\triangle$ACZ : $\triangle$BCZ,
and AZ : ZB :: $\triangle$AOZ : $\triangle$BOZ;
$\therefore$ AZ : ZB :: $\triangle$COA : $\triangle$BOC.
Similarly CY : YA :: $\triangle$BOC : $\triangle$AOB,



and BX : XC :: $\triangle$AOB : $\triangle$COA;
$\therefore$ AZ.CY.BX : ZB.YA.XC :: $\triangle$COA : $\triangle$COA;
$\therefore$ AZ.BX.CY = ZB.XC.YA.

The student should vary the figure by taking O outside $\triangle$ABC.

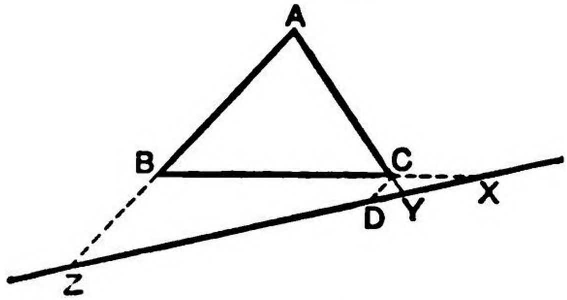
PROP. 2.—If the sides BC, CA, AB of a triangle ABC, produced or not, meet a straight line in the points X, Y, Z, then AZ.BX.CY = ZB.XC.YA (Menelaus's Theorem).

Draw CD $\parallel$ to AB to meet XYZ in D.
Then BX : XC :: ZB : CD

and CY : YA :: CD : AZ;

$\therefore$ AZ.BX.CY : ZB.XC.YA :: AZ.ZB.CD : ZB.CD.AZ;

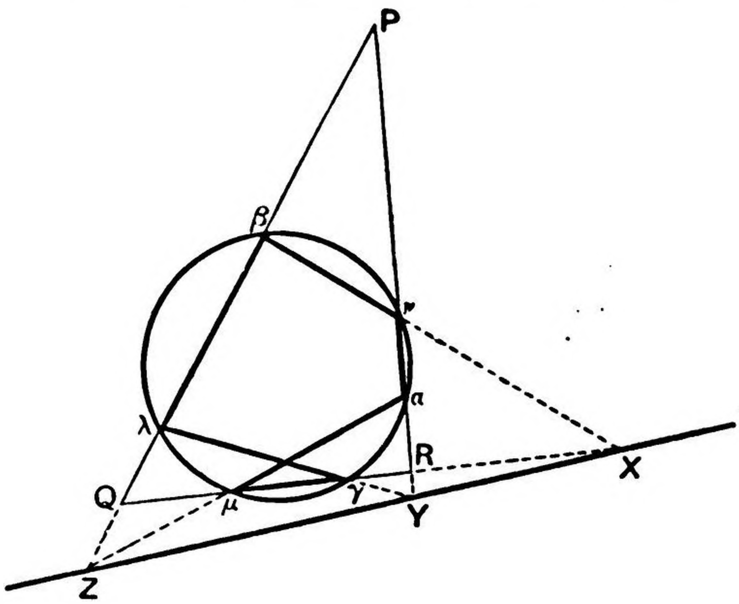
$\therefore$ AZ.BX.CY = ZB.XC.YA.



Ex. 892.—If X, Y, Z divide the sides BC, CA, AB of $\triangle$ ABC, so that AZ.BX.CY = ZB.XC.YA, then either AX, BX, CZ are concurrent, or X, Y, Z are collinear.

PROP. 3.—**If α, β, γ, λ, μ, ν are any six concyclic points, and if $\beta\nu$, $\mu\gamma$ intersect in X; $\gamma\lambda$ and $\nu\alpha$ in Y; and $\alpha\mu$, $\lambda\beta$ in Z; then X, Y, Z are collinear** (Pascal's Theorem).

Let PQR be $\triangle$ formed by $\beta\lambda$, $\gamma\mu$, $\alpha\nu$.



By three applications of Prop. 2, show that Pβ.QX.Rν.Pλ.Qγ.RY.PZ.Qμ.Rα = Qβ.RX.Pν.Qλ.Rγ.PY.QZ.Rμ.Pα.

Then remembering that $P\beta.P\lambda = P\nu.P\alpha$, apply Prop. 4 on Compound Ratio, and deduce QX.RY.PZ = RX.PY.QZ.

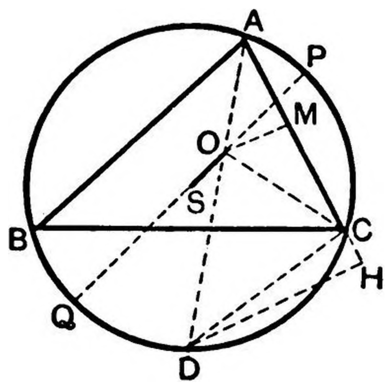
The pts. α, β, γ, λ, μ, ν can be taken in any order on the ⊙.

Ex. 893.—If a hexagon ABCDEF circumscribe a ⊙, the diagls. AD, BE, CF meet in a pt. (Brianchon's Theorem).

The pts. of contact α, β, γ, λ, μ, ν can be so taken that AD, BE, CF are the poles of the X, Y, Z of Prop. 3. *Use Ex.* 470.

PROP. 4.—If S and O be the circum- and in-centre R, r, the circum- and in-radius $SO^2 = R^2 - 2Rr$.

Let the in-⊙ touch AC at M: produce AO to cut the circum-⊙ at D: draw DH ⊥r to AC. Then it can be shown that OD = DC.



Again △s AOM, ADH are equiangr.;

∴ AO : AD :: OM : DH;

∴ AO.DC : AD.DC :: 2 R.OM : 2 R.DH.

But AD.DC = 2 R.DH. ['VI. C.'

2 R.OM = AO.DC

= AO.OD

$= R^2 - SO^2$; See Ex. 476.

∴ $SO^2 = R^2 - 2Rr$.

Ex. 894.—If O_1 be an ex-centre, and r_1 the corresponding ex-radius, $SO_1^2 = R^2 + 2Rr_1$.

Ex. 895.—If S, O be the centres, R, r the radii of two ⊙s, and $SO^2 = R^2 - 2Rr$, any number of △s can be both inscribed in one and circumscribed about the other.

PROP. 5.—Through a given point D to draw a straight line MDN which shall cut off from two given straight lines a triangle AMN equal to a given rectilineal figure.

Through D draw DF ∥ to AM to meet AN in F. To AF apply ∥gm AFGK equal to given rectl. figure. At K erect ⊥r KL equal to FD, and describe ⊙ with centre L and radius equal to DG, cutting AM at M. Join MD and produce to cut AN at N.

Then FD, DG are homols. sides of simr. △s FDN, DGH;

$$\therefore \triangle FDN : \triangle HDG :: FD^2 : DG^2.$$

$$\text{Similarly } \triangle KMH : \triangle HDG :: KM^2 : DG^2;$$

$$\therefore \triangle FDN + \triangle KMH : \triangle HDG :: FD^2 + KM^2 : DG$$

$$:: KL^2 + KM^2 : LM^2$$

$$\therefore \triangle FDN + \triangle KMH = \triangle HDG;$$

$$\therefore \triangle AMN = \|\text{gm } AG.$$

The above investigation may be adapted to the case in which D is outside ∠ MAN.

Lemma.—D is any fixed point; bc a fixed straight line, touching a fixed circle at P; L is any other point on bc; along DL is taken DE such that DL.DE = sq. of tangent from D to the fixed circle; then E lies on another fixed circle touching the first and passing through D.

Let DP cut the first ⊙ again in Q, and let QH be the tangt. at Q.

Then DL.DE = DP.DQ;

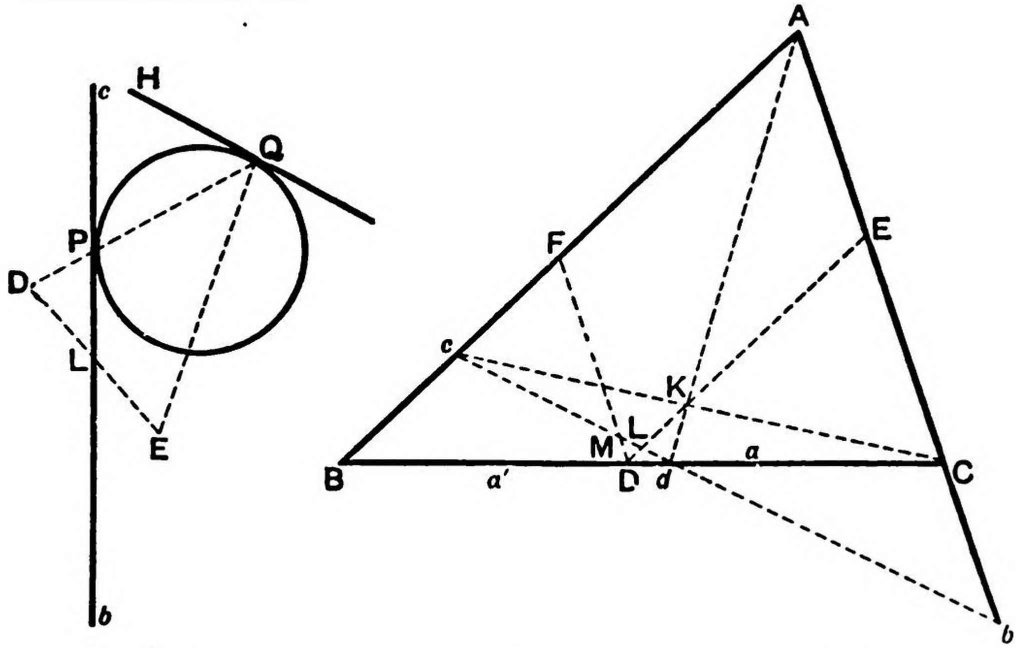
∴ E, L, P, Q are concyclic;

∴ ∠ DEQ = ∠ cPQ = ∠ HQP;

∴ E lies on a fixed ⊙ through D and Q, touching HQ, and ∴ the first fixed ⊙ at Q.

PROP. 6.—The nine-point circle of a triangle touches the in- and ex-circles.

Consider the in-$\odot$ and that ex-$\odot$ which touches BC between B and C.



BC, CA, AB are three of the common tangts. to these two $\odot$s. Let the fourth common tangt. bdc be drawn cutting AC, CB, BA in b, d, c. Let D, E, F be mid pts. of BC, CA, AB. Then Ab=AB; Ac=AC, and Ad is the intl. bisector of $\angle$ BAC;

$\therefore$ Ad bisects Cc at rt. $\angle$s,

and $\therefore$ DE passes through K, the pt. where Ad cuts Cc.

Also if a, a' be the pts. of contact with BC,

$$Da' = Da = \tfrac{1}{2}(BA - AC) = \tfrac{1}{2}Bc = DK.$$

Let DE, DF cut bc in L, M.

Then $DL : DK :: Bc : BA :: DK : DE$;

$$\therefore DL.DE = DK^2 = Da^2 = Da'^2.$$

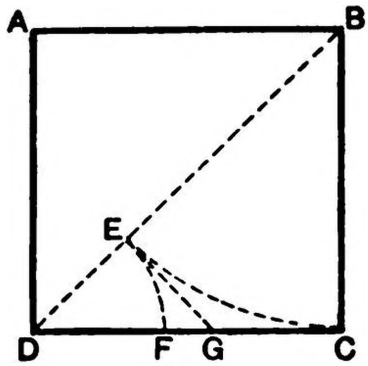
Similarly $DM.DF = Da^2 = Da'^2$;

$\therefore$ E and F lie on the $\odot$ through D touching the in- and ex-$\odot$s considered.

PROP. 7.—**The diagonal and the side of a square have no common measure.**

From the diagl. BD of the sq. ABCD cut off BE equal to BC and draw EG ⊥r to BD to meet CD in G.

Then DE=EG=GC and DE : DG :: BC : BD.



Any line which measured BC and BD would also measure DE, and ∴ also GC.

But since it measured CD it would measure DG, and ∴ be a common measure of DE and DG.

Again ∵ DE : DG :: BC : BD and DE, DG < BC, BD.

G.C.M of DE, DG < G.C.M. of BC, BD.

But we have shown that any line which measured BC, BD would also measure DE, DG.

MISCELLANEOUS EXERCISES.

Books I.-VI.

Ex. 896.—In a $\triangle ABC$, $BC = 2\,AC$ and $\angle A = 3\,\angle B$. Show that $\triangle ABC$ is right angled.

Ex. 897.—ABCD is a parallelogram; P any point within it. Show that $\triangle ABP + \triangle CDP = \triangle BCP + \triangle DAP$.

Ex. 898.—AB, AC are two given straight lines, and X and Y any other two straight lines not parallel to one another. Show how to draw a straight line parallel to X such that the intercept made upon it by AB, AC shall be bisected by Y.

Ex. 899.—ABC is an isosceles triangle, right-angled at C. The internal bisector of angle BAC meets BC in D. Show that $CD = AB - AC$.

Ex. 900.—Given any point in one of the shorter sides of an oblong. Show that two rectangles may be constructed which have an angle at the given point and which are inscribed in the oblong.

Ex. 901.—If the base AB of a triangle be divided at D so that $m.AD = n.DB$, show that $m.AC^2 + n.BC^2 = m.AD^2 + n.BD^2 + (m+n)CD^2$.

Ex. 902.—O, C are the mid points of the arcs into which a chord AB divides a given circle, and PAQ is the tangent to the circle at A. Show that AO, AC are the bisectors of the angles BAP, BAQ.

Of what general Proposition is this a limiting case?

Ex. 903.—In the fig. of 'VI. B,' if the tangent at A meet BC produced in T; show that $TA = TD$.

Ex. 904.—AB is a fixed diameter of a given circle APB; P any point on the circumference. If the tangent at P meet the tangents at A and B in Q and R; show that the rectangle PQ, PR is equal to the square of the radius of the circle. Show also that the rectangle AQ, BR is equal to the square of the radius.

Ex. 905.—A, B, C, D, E, F are four points on a circle. If DE is parallel to AC and CF parallel to BD; show that EF is parallel to AB.

Ex. 906.—A tangent drawn at a point on that part of the in-circle

convex toward A, meets AB, AC in D, E. Prove that the difference of the perimeters of triangles ABC, ADE = twice BC.

Ex. 907.—Deduce the existence of Simson's line from Ptolemy's theorem. Also deduce Ptolemy's theorem and the extension to it from the existence of Simson's line.

Ex. 908.—AD, BE, CF are the perpendiculars from A, B, C to the sides BC, CA, AB of triangle ABC. Show that EF is perpendicular to the diameter through A of the circum-circle of triangle ABC.

Ex. 909.—One of the sides containing the right angle of a right-angled triangle is double the other, and circles are described on these sides as diameters. Show that their common chord is two-fifths of the hypotenuse.

Ex. 910.—From A in the acute-angled triangle ABC is drawn AD perpendicular to BC. If AD = 6, BD = 3, CD = 2, show that angle BAC is half a right angle. *Use Exx.* 427, 429.

Ex. 911.—In triangle ABC the angle C is obtuse and AD is perpendicular to BC. If AD = 2 or 3, BD = 6, CD = 1, show that angle BAC is half a right angle.

Ex. 912.—If circles be described, each of which touches a pair of alternate sides of a regular pentagon at the extremities of the intermediate side, show that all such circles have a common point.

Ex. 913.—If ABCDE be any cyclic pentagon and the chords BD, CE meet in a; CE, DA in b; and so on, so that a second pentagon abcde is formed, show that

$$\angle BAE + \angle bae = \angle ABC + \angle AED.$$

Ex. 914.—In a circle APQ, centre C, two parallel chords PMP′, QNQ′ are drawn bisected by the radius CNMA. If the lines QP, QP′ cut CA in U, V, prove that $CU, CV = CA^2$.

Ex. 915.—The straight lines joining the ends of any chord of a circle to the mid point of a chord through its pole are equally inclined to the latter chord.

Ex. 916.—A series of triangles is formed in the following manner :— The ex-centres of each are the vertices of the next. Show that as we proceed the triangles tend to become equilateral.

Ex. 917.—L is the point of contact of the in-circle of triangle ABC

with BC. Show that one of the common tangents of the two circles described through L to touch AB at B and AC at C is parallel to BC.

Ex. 918.—O is the in-centre of triangle ABC; P the point where AO meets the in-circle: BO, CO are produced to meet the tangent at P in H, K. Show that BH=CK=BO+OC. Also if the same tangent meets AB, AC in L, M; show that CM=HL and that BL=KM (Leybourn's *Mathematical Repository*).

Ex. 919.—A triangle APB has a fixed base AB and angle B exceeds angle A by a constant difference. Show that the tangent at P to the circum-circle of triangle APB is fixed in direction.

Ex. 920.—If the sum or difference of the two base angles of a triangle be equal to a right angle, prove that the perpendicular upon the base from the third angle is a mean proportional between the segments of the base.

Ex. 921.—The tangent at A to the circum-circle of triangle ABC meets BC produced at D; BA produced meets the circum-circle of triangle ACD at E; and the tangent at B to circum-circle of triangle ABC meets the circum-circle of triangle ABD again at F. Show that BEDF is a parallelogram.

Ex. 922.—In the fig. of VI. 13, if a chord AHK be drawn cutting BD in H, show that AD touches the circum-circle of triangle DHK.

Ex. 923.—Area of regular 12-gon=3 times square on radius of circum-circle.

Ex. 924.—ABCD is a rectangle, and E a point in CD such that AE is a mean proportional between AB, BC. If BF be drawn perpendicular to AE, show that BF=AE.

Hence show how to cut a rectangular card whose length is not greater than twice its breadth into three parts, which may be put together to form a square. Examine the case in which the length is double the breadth, and point out why the construction fails when the length is more than double the breadth.

Ex. 925.—Use the last exercise to suggest a method of dividing a rectangle whose length is greater than twice its breadth into four parts which may be put together to form a square.

Hence obtain a demonstration of I. 47, showing that CL can be cut into pieces equivalent to AK, and BL into pieces equivalent to AF.

Ex. 926.—Given the vertical angle, a line drawn from thence to divide

the base in a given ratio and the sum of the squares on the other two sides, to construct the triangle (Hutton's *Miscellanea Mathematica*).

Ex. 927.—The straight lines AO, BO, CO, joining the vertices of triangle ABC to any point O, are produced to cut the circum-circle again in P, Q, R; D, E, F are the projections of O on BC, CA, AB. Show that triangles PQR, DEF are directly similar, and that O in PQR corresponds to the point in DEF isogonally inverse to O. *See Ex.* 439.

Ex. 928.—If O in Ex. 927 is the symmedian point of *either* of the triangles ABC, PQR, it is also the symmedian point of the other, and that the sides of each triangle are proportional to the medians of the other.

Ex. 929.—If O in Ex. 927 is the positive Brocard point of triangle ABC, it is the negative Brocard point of triangle DEF.

Ex. 930.—If O in Ex. 927 is the centroid of the triangle ABC, and $A\alpha$, $B\beta$, $C\gamma$ are symmedian chords, then $A\alpha . QR = B\beta . RP = C\gamma . PQ$.

Ex. 931.—ABC..., A′B′C′... are two directly similar figures, I their centre of stretch-rotation. ABC... can be transformed into A′B′C′... by

(1) a 'turn' ω, about any point in its plane, followed by a 'stretch' from a centre S;

(2) a 'stretch' from S, followed by a 'turn' ω, about a certain point O′ not coincident with O. Show that

(3) S, O, O′ are collinear.

(4) SO is stretched into SO′.

(5) IO, IO′ are equally inclined to SI, and that

$$IO : IO' :: SO : SO'.$$

(6) O′ is the position of O after the first 'turn and stretch' (Prof. R. W. Genese.)

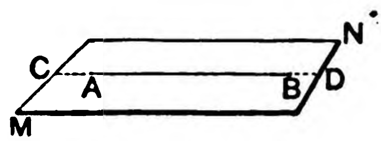
THE HARPUR EUCLID

BOOK XI. (1-21).

BOOK XI.

PROP. 1.—**One part of a straight line cannot be in a plane and another part without it.**

Let the segment AB of a st. line lie in pl. MN; then the whole st. line lies in pl. MN.



For AB can be produced to any length, as CD, in pl. MN.

And if AB were a segt. of any other st. line ABK other than CD, the pl. MN could be turned about CD until the pt. K, and ∴ the st. line ABK, fell in it; [I. Def. 7.

I.e. two st. lines in a plane would have a common segment, which is impossible.

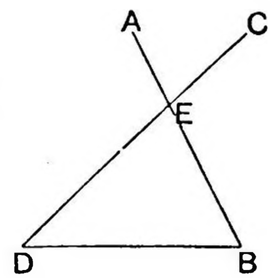
Note.—*Simson gave a faulty demonstration (which we have omitted) of the proposition that 'two straight lines in a plane cannot have a common segment' as a corollary to I.* 11.

The reason given by Euclid is that if two straight lines ABK, ABD *had a common segment* AB, *and a circle were described with centre* B *and radius* AB, *the diameters* ABK, ABD *would cut off unequal arcs from that circle.*

PROP. 2.—(i) Two straight lines which cut one another are in the same plane.

(ii) Three straight lines which meet one another are in the same plane.

Let the st. lines AB, CD cut one another at E, then they are in the same plane. Also the three st. lines BD, DE, EB are in the same plane.



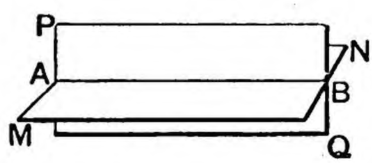
Let any plane pass through EB and be turned about EB, produced if necessary, until it passes through D. Then BD, DE are in that plane; [I. Def. 7.

∴ BD, DE, EB are in one plane;

∴ also AB, CD are in that same plane, XI. 1.

PROP. 3.—If two planes cut one another, their common section is a straight line.

Let the pls. MN, PQ cut one another; their common section, AB, shall be a st. line.



For the st. line from A to B lies in both planes; [I. Def. 7.

∴ it is their common section.

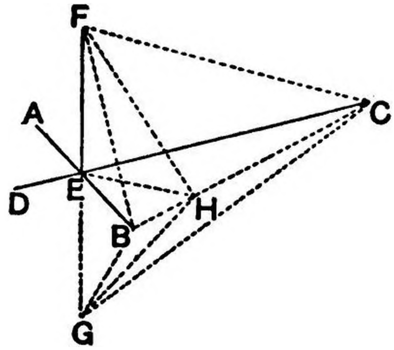
Def.—**A straight line is said to be perpendicular or at right angles to a plane when it is at right angles to all straight lines meeting it in that plane.**

PROP. 4.—If a straight line is at right angles to each of two straight lines at their point of intersection, it is at right angles to the plane in which they are.

Let the st. line EF be ⊥r to each of the two st. lines AB,

CD, cutting each other at E; then EF is ⊥r to the plane in which they are.

Join BC. In BC take any pt. H. Produce FE to G so that EG=EF. Join FB, FH, FC, GB, GH, GC, EH.



In △s FEB, GEB $\begin{cases}\text{FE, EB=GE, EB,}\\ \text{and rt. ∠ FEB=rt. ∠ GEB;}\end{cases}$

∴ FB=GB.

Similarly FC=GC.

But BC is common to △s FBC, GBC;

∴ ∠FBH=∠GBH.

But FB, BH=GB, BH;

∴ FH=GH.

But FE, EH=GE, EH;

∴ ∠FEH=∠GEH;

∴ they are rt. ∠s;

∴ FE is ⊥r to EH.

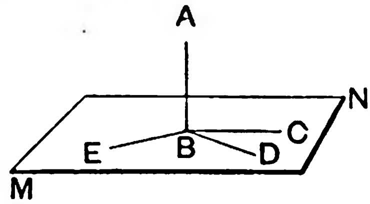
Similarly it can be shown ⊥r to any other st. line, meeting it in pl. BEC;

∴ FE is ⊥r to pl. containing AB, CD.

PROP. 5.—**If three straight lines meet at a point, and a straight line stand at right angles to each of them at that point, these three straight lines are in one and the same plane.**

Let the st. line AB be ⊥r to each of the st. lines BC, BD, BE; then BC, BD, BE are in the same plane.

Let pl. MN be that in which lie BD, BE. Then the pl. containing AB, BC cuts pl. MN in some st. line (XI. 3) which is ⊥r to AB (XI. 4);

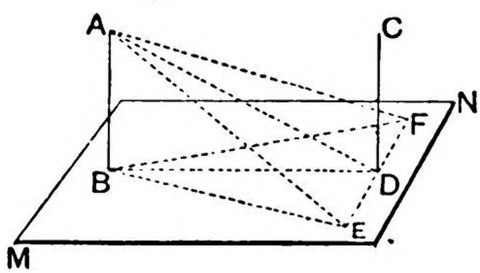


∴ it must lie along BC.
i.e. BC is in the same plane with BD, BE.

PROP. 6.—**If two straight lines are at right angles to the same plane, they are parallel to one another.**

Let AB, CD be ⊥r to pl. MN,
then AB is ∥ to CD.

Join BD. Through D in pl. MN draw EDF ⊥r to BD such that ED=DF. Join BE, BF, AD, AE, AF.



In △s BDE, BDF $\begin{cases} \text{BD, DE=BD, DF,} \\ \text{and rt. ∠ BDE=rt. ∠ BDF;} \end{cases}$

∴ BE=BF.

Again ABE, ABF are rt. ∠s (∵ AB is ⊥r to pl. MN);

∴ in △s ABE, ABF $\begin{cases} \text{AB, BE=AB, BF,} \\ \text{and ∠ ABE= ∠ ABF;} \end{cases}$

∴ AE=AF.
But AD, DE=AD, DF;
∴ ∠ ADE= ∠ ADF;
∴ AD is ⊥r to EF.

But BD is $\perp$r to EF, [CONST.

and CD is $\perp$r to EF ($\because$ CD is $\perp$r to pl. MN);

$\therefore$ CD is in same plane with AD, DB; [XI. 5.

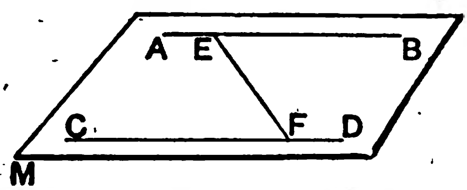
$\therefore$ CD is in same plane with AB. [XI. 2.

But $\angle$s ABD, CDB are rt. $\angle$s ($\because$ AB, CD are $\perp$r to pl. MN);

$\therefore$ AB, CD are $\parallel$.

PROP. 7.—**If two straight lines be parallel, the straight line drawn from any point in one to any point in the other is in the same plane with the parallels.**

Let AB, CD be two $\parallel$ st. lines; E, F any two pts. in AB, CD; then EF lies in the pl. MN containing AB, CD.



For E, F lie in pl. MN;

$\therefore$ EF lies in that plane. [I. DEF. 7.

PROP. 8.—**If two straight lines be parallel, and one of them be at right angles to a plane, the other shall also be at right angles to the same plane.**

Let the st. lines AB, CD be $\parallel$, and let AB be $\perp$r to pl. MN; then CD is $\perp$r to pl. MN. Join BD. Through D, in pl. MN, draw EDF $\perp$r to BD such that ED=DF. Join BE, BF, AD, AE, AF.

In $\triangle$s BDE, BDF $\begin{cases} \text{BD, DE=BD, DF,} \\ \text{and rt. } \angle \text{ BDE=rt. } \angle \text{ BDF;} \end{cases}$

$\therefore$ BE=BF.

Again ABE, ABF are rt. $\angle$s ($\because$ AB is $\perp$r to pl. MN);

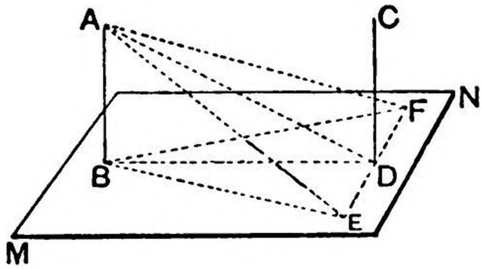
$\therefore$ in $\triangle$s ABE, ABF $\begin{cases} \text{AB, BE=AB, BF,} \\ \text{and } \angle \text{ ABE=} \angle \text{ ABF;} \end{cases}$

$\therefore$ AE=AF.

But AD, DE = AD, DF

$\therefore$ $\angle$ ADE = $\angle$ ADF ;

$\therefore$ ED is $\perp$r to AD.



But ED is $\perp$r to BD ; [Const.

$\therefore$ ED is $\perp$r to pl. ADB ; [XI. 4.

$\therefore$ ED is $\perp$r to CD, which is in pl. ADB. [XI. 7.

Again $\because$ $\angle$s ABD, BDC = two rt. $\angle$s,

and $\angle$ ABD is a rt. $\angle$ ($\because$ AB is $\perp$r to pl. MN) ;

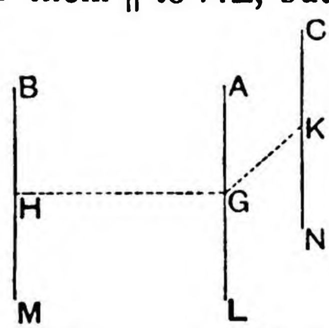
$\therefore$ CDB is a rt. $\angle$.

But CDE is a rt. $\angle$;

$\therefore$ CD is $\perp$r to pl. MN. [XI. 4.

PROP. 9.—**Two straight lines which are each of them parallel to the same straight line and not in the same plane with it, are parallel to one another.**

Let the st. lines BM, CN be each of them || to AL, but not in the same plane with it ; BM shall be || to CN.



From any pt. G in AL and $\perp$r to AL, draw GH in the plane containing AL, BM, and GK in the plane containing AL, CN.

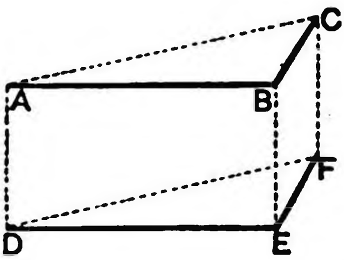
Then AL is $\perp$r to pl. HGK ; [XI. 4.

$\therefore$ BM and CN are $\perp$r to pl. HGK ; [XI. 6.

$\therefore$ MB is || to CN. [XI. 8.

PROP. 10.—If two straight lines which meet one another be parallel to two other straight lines which meet one another and are not in the same plane with the first two, the first two and the other two shall contain equal angles.

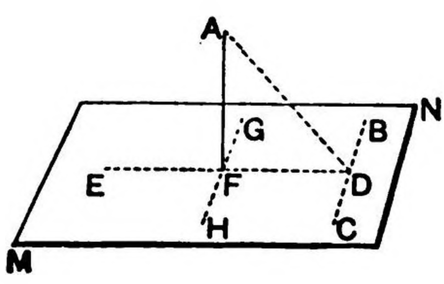
Let the st. lines AB, BC be ∥ to the st. lines DE, EF; then ∠ ABC = ∠ DEF. Take BA, BC, ED, EF all equal to one another. Join AD, BE, CF, AC, DF.



∵ AB = DE and is ∥ to DE;
∴ AD = BE and is ∥ to BE.
Similarly CF = BE and is ∥ to BE;
∴ AD = CF and is ∥ to CF;
∴ AC = DF.
But AB, BC = DE, EF;
∴ ∠ ABC = ∠ DEF.

PROP. 11.—To draw a straight line perpendicular to a plane from a given point without it.

Let A be a given pt. without the pl. MN; it is reqd. to draw from A a st. line ⊥r to pl. MN. In pl. MN draw any st. line BC. From A draw AD ⊥r to BC.



Then if AD s not ⊥r to pl. MN, in pl. MN draw DE ⊥r to BC, and from A draw AF ⊥r to DE. AF is the reqd. ⊥r to pl. MN.

Through F draw in pl. MN GFH || to BC ;

∵ BC is ⊥r to AD, DF ;

∴ BC is ⊥r to pl. ADE ; [XI. 4.

∴ GH is ⊥r to pl. ADE ; [XI. 8.

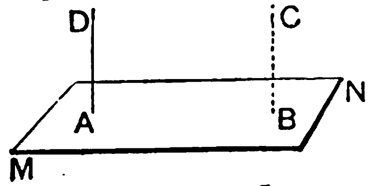
∴ GH is ⊥r to AF ;

∵ AF is ⊥r to GH, DE ;

∴ AF is ⊥r to pl. MN. [XI. 4.

PROP. 12.—To erect a straight line at right angles to a given plane from a given point in the plane.

Let A be the given pt., MN the given plane ; it is reqd. to erect a st. line from A ⊥r to pl. MN.



From any pt. C without pl. MN draw CB ⊥r to pl. MN.

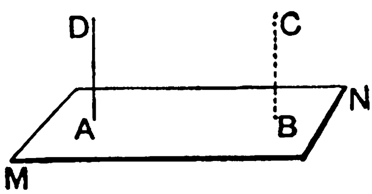
From A draw AD || to CB.

Then AD is ⊥r to pl. MN. [XI. 8.

PROP. 13.—(1) From the same point in a given plane there cannot be drawn two straight lines at right angles to the plane.

(2) From a given point without a plane there can only be drawn one perpendicular to the plane.

Let A be a pt. in pl. MN ; only one ⊥r can be drawn to pl. MN from A.



Take any other pt. B in pl. MN and erect at B a ⊥r to pl. MN.

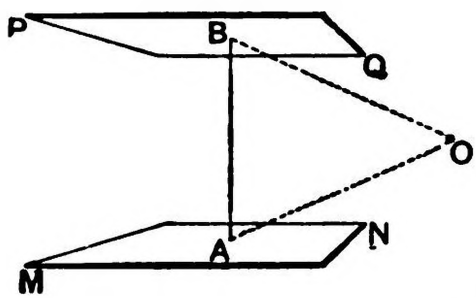
Then any ⊥r from A to pl. MN must be || to BC (XI. 6), and if two could be drawn we should have two st. lines || to BC and meeting at A, which is impossible (I. 30 and XI. 9).

(2) If two ⊥rs could be drawn to pl. MN from an extl. pt. A, they would be || to one another (XI. 6), which is absurd.

DEF.—**Parallel planes are such as do not meet one another though produced.**

PROP. 14.—**Planes to which the same straight line is perpendicular are parallel to one another.**

Let the st. line AB be ⊥r to each of the pts. MN, PQ; then pl. MN is ‖ to pl. PQ.



For, if possible, let O be a pt. common to pls. MN, PQ when produced. Join OA, OB.

Then ∵ AB is ⊥r to pls. MN, PQ,

∴ ∠s OBA, OAB are rt. ∠s,

which is impossible;

∴ pl. MN is ‖ to pl. PQ.

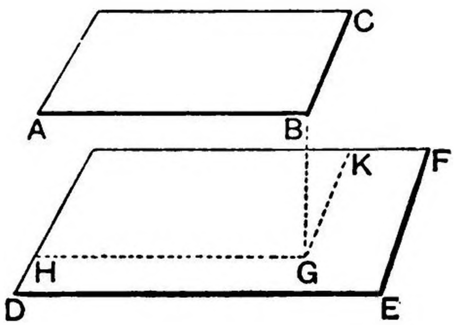
PROP. 15.—**If two straight lines which meet one another be parallel to two other straight lines which meet one another, but are not in the same plane with the first two, the plane which passes through these is parallel to the plane which passes through the others.**

Let the st. line AB, BC be ‖ to the st. lines DE, EF, but not in the same plane with them; then pl. ABC is ‖ to pl. DEF.

From B draw BG ⊥r to pl. DEF,

and through G in pl. DEF draw GH, GK || to DE, EF, and
∴ also || to AB, BC.

Then ∠s ABG, BGH = 2 rt. ∠s.



But ∠ BGH is a rt. ∠ (∵ BG is ⊥r to pl. DEF);

∴ ∠ ABG is a rt. ∠.

Similarly CBG is a rt. ∠;

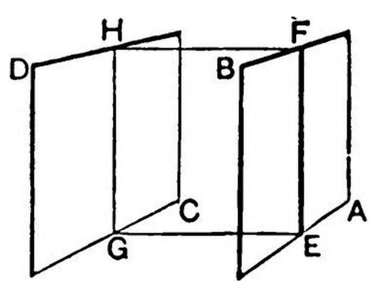
∴ BG is ⊥r to pl. ABG. [XI. 4.

But BG is ⊥r to pl. DEF;

∴ pl. ABC is || to pl. DEF. [XI. 14.

PROP. 16.—**If two parallel planes be cut by a third plane, their common sections with it are parallels.**

Let the || pls. AB, CD be cut by pl. EFHG, their common sections EF, GH shall be ||.

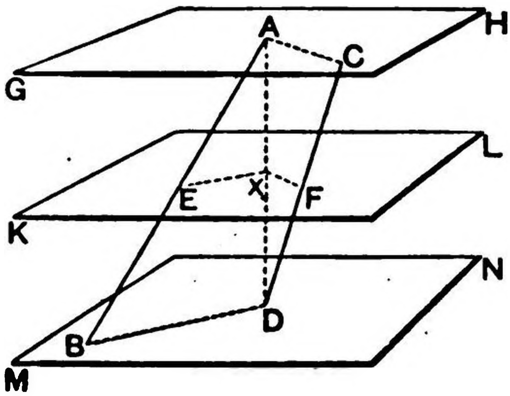


For EF, GH lie in the same pl. EFGH, and if they met when produced, the pls. AB, CD, in which they lie, would also meet, which is contrary to the hypothesis.

PROP. 17.—**If two straight lines be cut by parallel planes, they shall be cut in the same ratio.**

Let the st. lines AB, CD be cut by the || pls. GH, KL, MN in the pts. A, E, B, C, F, D; then AE : EB :: CF : FD.

Join AC, BD, AD. Let AD cut pl. KL in X. Join EX, XF.



∵ pl. ADB cuts the || pls. KL, MN in EX, BD;
∴ EX is || to BD; [XI. 16.
∴ AE : EB :: AX : XD.
Similarly CF : FD :: AX : XD;
∴ AE : EB :: CF : FD.

DEF.—**A plane is perpendicular to a plane when the straight lines drawn in one of the planes perpendicularly to the common section of the planes are perpendicular to the other planes.**

PROP. 18.—**If a straight line be at right angles to a plane, every plane which passes through that line shall be at right angles to that plane.**

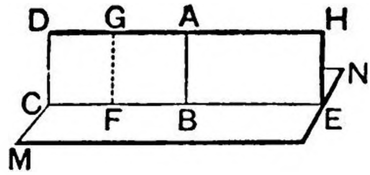
Let the st. line AB be ⊥r to pl. MN; then any pl. DE through AB is ⊥r to pl. MN.

From any pt. F in CE, the common section of the two pts., draw FG in pl. DE ⊥r to CE.

Then ABF is a rt. ∠ (∵ AB is ⊥r to pl. MN).

But BFG is a rt. ∠.

∴ FG is ∥ to AB;



But AB is ⊥r to pl. MN;

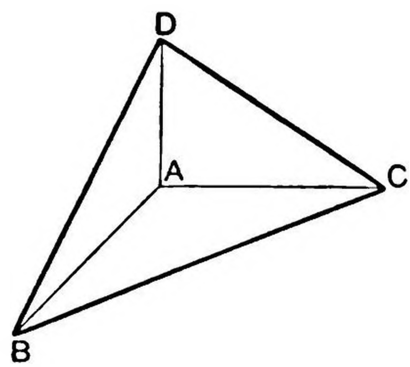
∴ FG is ⊥r to pl. MN. [XI. 8.

Similarly any other st. line in pl. DE ⊥r to CE is ⊥r to pl. MN;

∴ pl. DE is ⊥r to pl. MN.

PROP. 19.—**If two planes which cut one another be each of them perpendicular to a third plane, their common section shall also be perpendicular to the same plane.**

Let the two pls. ABD, ACD be each of them ⊥r to pl. ABC; then their common section AD shall be ⊥r to pl. ABC.



∵ pl. BAD is ⊥r to pl. BAC;

∴ the ⊥r to pl. BAC through A lies in pl. BAD. [Def.

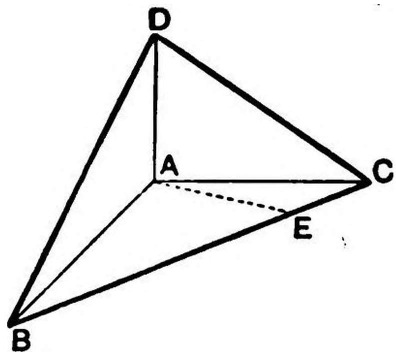
Similarly it lies in pl. CAD;

∴ it must be their common section AD.

Def.—A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, at the same point.

PROP. 20.—**If a solid angle be contained by three plane angles, any two of them are greater than the third.**

Let the solid ∠ at A be contained by the three plane ∠s BAC, CAD, DAB; any two of them are greater than the third. If they are not all equal, let ∠ BAC > ∠ BAD and $\geqq$ ∠ CAD.



In pl. BAC draw AE such that ∠ BAE = ∠ BAD and AE = AD.

Through E draw the st. line BEC. Join DB, DC.

In △s BAD, BAE $\begin{cases} \text{BA, AD} = \text{BA, AE,} \\ \text{and } \angle \text{BAD} = \angle \text{BAE;} \end{cases}$

∴ BD = BE.

But BD, DC > BC;

∴ remr. DC > remr. EC.

In △s CAD, CAE $\begin{cases} \text{CA, AD} = \text{CA, AE,} \\ \text{but DC} > \text{EC;} \end{cases}$

∴ ∠ CAD > ∠ CAE.

But ∠ BAD = ∠ BAE.

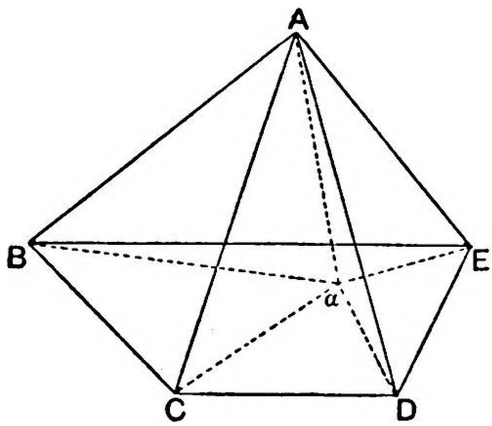
∴ ∠s BAD, CAD > whole ∠ BAC.

But ∠ BAC > BAD and $\geqq$ CAD;

∴ ∠ BAC with either ∠ BAD or ∠ CAD > the other.

PROP. 21.—Every solid angle is contained by plane angles which are together less than four right angles.

Let the solid ∠ at A be contained by any number of plane ∠s BAC, CAD, DAE, EAB; these are together less than four rt. ∠s.



Let the pls. in which these ∠s are be cut by a plane BCDE in the st. lines BC, CD, DE, EB.

Take any pl. a within BCDE and join it with the angular pts., thus dividing it into as many △s as it has sides, *i.e.* as there are plane ∠s containing A.

Then ∠s ACB, ACD > ∠ BCD.

∴ ∠s ACB, ACD > ∠s aCB, aCD,

and so on at D, E, B;

∴ ∠s at bases of △s with common vertex at A, are greater than ∠s at bases of △s with common vertex at a; but all ∠s of first of △s = all ∠s of second set. [I. 32.

∴ remg. ∠s at A < remg. ∠s at a;

∴ ∠s at A < four rt. ∠s.

DEFINITIONS.

Book XI.

1. A 'solid' is that which has length, breadth, and thickness.

2. That which bounds a solid is a 'superficies.'

3. A straight line is 'perpendicular,' or 'at right angles,' to a plane, when it makes right angles with every straight line meeting it in that plane.

4. A plane is 'perpendicular,' or at right angles, to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

5. The 'inclination of a straight line to a plane' is the acute angle contained by that straight line, and another drawn from the point at which the first line meets the plane to the point at which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

6. The 'inclination of a plane to a plane' is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one in one plane, and the other in the other plane.

7. Two planes are said to have the 'same or a like inclination' to one another, which two other planes have, when the said angles of inclination are equal to one another.

8. 'Parallel planes' are such as do not meet one another though produced.

9. A 'solid angle' is that which is made by more than two plane angles, which are not in the same plane, meeting at one point.

10. 'Equal and similar solid figures' are such as are contained by similar planes equal in number and magnitude.

11. 'Similar solid figures' are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

12. A 'pyramid' is a solid figure contained by planes which

are constructed between one plane and one point above it at which they meet.

13. A 'prism' is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others are parallelograms.

14. A 'sphere' is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed.

15. The 'axis of a sphere' is the fixed straight line about which the semicircle revolves.

16. The 'centre of a sphere' is the same with that of the semicircle.

17. The 'diameter of a sphere' is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

18. A 'cone' is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled cone; and if greater, an acute-angled cone.

19. The 'axis of a cone' is the fixed straight line about which the triangle revolves.

20. The 'base of a cone' is the circle described by that side containing the right angle which revolves.

21. A 'cylinder' is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

22. The 'axis of a cylinder' is the fixed straight line about which the parallelogram revolves.

23. The 'bases of a cylinder' are the circles described by the two revolving opposite sides of the parallelogram.

24. 'Similar cones and cylinders' are those which have their axes and the diameters of their bases proportionals.

25. A 'cube' is a solid figure contained by six equal squares.

26. A 'tetrahedron' is a solid figure contained by four equal and equilateral triangles.

27. An 'octahedron' is a solid figure contained by eight equal and equilateral triangles.

28. A 'dodecahedron' is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

29. An 'icosahedron' is a solid figure contained by twenty equal and equilateral triangles.

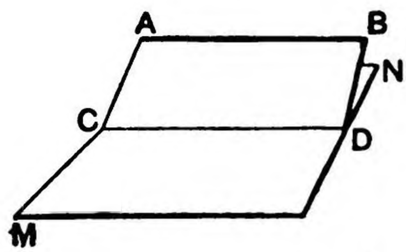
A. A 'parallelopiped' is a solid figure contained by six quadrilateral figures, of which every opposite two are parallel.

The following, though not given by Euclid, are important (Legendre's *Éléments de Géométrie*) :—

Def.—A straight line and a plane are said to be parallel to one another when they do not meet however far they are produced.

PROP. 1.—If a straight line AB without a plane MN be parallel to a straight line CD in that plane, it is parallel to the plane.

For, since it lies in pl. ABDC, if it met pl. MN it would



meet it at some point in the common section CD, which is contrary to the hypothesis.

PROP. 2.—If a straight line AB is parallel to a plane MN, and a plane passing through AB cut the plane MN, the common section CD is parallel to AB.

For if AB met CD it would meet pl. MN, in which CD lies, which is contrary to the hypothesis.

PROP. 3.—If from a point C in a plane MN, which is parallel to a straight line AB, a straight line be drawn parallel to AB, it will lie in the plane MN.

For the plane BAC cuts pl. MN in a st. line CD || to AB by Prop. 2, and only one || to AB can be drawn from a point. [I. 30 and XI. 9.

PROP. 4.—If a straight line AB is parallel to two planes CDE, CDF, which cut one another, it is parallel to the common section CD.

For the || to AB through C must lie in both planes by Prop. 3;

∴ it is their common section CD.

LIST OF MATHEMATICAL TERMS USED IN THIS WORK, BUT NOT DEFINED BY EUCLID.

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